A HISTORY OF MATHEMATICAL NOTATIONS

FLORIAN CAJORI

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Volume II

NOTATIONS MAINLY IN
HIGHER MATHEMATICS

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PREFACE TO THE SECOND VOLUME

The larger part of this volume deals with the history of notations in higher mathematics. The manuscript for the parts comprising the two volumes of this History was completed in August, 1925, but since then occasional alterations and additions have been made whenever new material or new researches came to my notice.

Some parts of this History appeared as separate articles in scientific and educational journals, but later the articles were revised and enlarged.

I am indebted to Professor R. C. Archibald and to Professor L. C. Karpinski for aid in the arduous task of reading the proofs of this volume.

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INTRODUCTION TO THE SECOND VOLUME

It has been the endeavor to present in the two volumes of this *History* a fairly complete list of the symbols of mathematics down to the beginning of the nineteenth century, and a fairly representative selection of the symbols occurring in recent literature in pure mathematics. That we have not succeeded in gathering all the symbols of modern mathematics is quite evident. Anyone hunting, for even an hour, in the jungle of modern mathematical literature is quite certain to bag symbolisms not mentioned in this *History*. The task of making a complete collection of signs occurring in mathematical writings from antiquity down to the present time transcends the endurance of a single investigator. If such a history were completed on the plan of the present work, it would greatly surpass this in volume. At the present time the designing of new symbols is proceeding with a speed that is truly alarming.

Diversity of notation is bound unnecessarily to retard the spread of a knowledge of the new results that are being reached in mathematics. What is the remedy? It is hoped that the material here presented will afford a strong induction, facilitating the passage from the realm of conjecture as to what constitutes a wise course of procedure to the realm of greater certainty. If the contemplation of the mistakes in past procedure will afford a more intense conviction of the need of some form of organized effort to secure uniformity, then this *History* will not have been written in vain.

ADDENDA

PAGE 28, line 3, *add the following:* In the Commonplace Book of Samuel B. Beach, B.A., Yale, 1805, now kept in the Yale University Library, there is given under the year 1804, "the annual expence about \$700," for the upkeep of the lighthouse in New Haven. The dollar mark occurs there in the conventional way now current. Prof. D. E. Smith found the symbol \$ very nearly in the present form in Daboll's *Schoolmaster's Assistant,* 4th edition, 1799, p. 20. Dr. J. M. Armstrong of St. Paul, Minn., writes that in the *Medical Repository,* New York (a quarterly publication), Vol. III, No. 3, November and December, 1805, and January, 1806, p. 312, the \$ is used as it is today.

PAGE 29, line 6, *add the following:* Since this volume was printed, important additional and confirmatory material appeared in our article "New Data on the Origin and Spread of the Dollar Mark" in the *Scientific Monthly,* September, 1929, p. 212–216.

PAGE 145, line 1, *for* Gioseppe *read* Giuseppe Moleti

PAGE 323, lines 8 and 9, *for* in G. Cramer . . . found earlier *read* in Claude Rabuel's *Commentaires sur la Géométrie de M. Descartes,* Lyons, 1730, and in G. Cramer . . . found also.

In the alphabetical index *insert* Mahnke, D., 542, 543, 563.

I

TOPICAL SURVEY OF SYMBOLS IN ARITHMETIC AND ALGEBRA (ADVANCED PART)

LETTERS REPRESENTING MAGNITUDES

388. *Greek period.*—The representation of general numbers by letters goes back to Greek antiquity. Aristotle uses frequently single capital letters, or two letters, for the designation of magnitude or number. For example, he says: "If A is what moves, B what is being moved, and Γ the distance over which it was moved, and Δ the time during which it was moved, then the same force A, in the same time could move the half of B twice as far as Γ, or also in half the time Δ exactly as far as Γ."[1] In other places[2] he speaks of the "$B\Gamma$ any force," "the time EZ." In another place he explains how much time and trouble may be saved by a general symbolism.[3]

Euclid[4] in his *Elements* represented general numbers by segments of lines, and these segments are marked by one letter,[5] or by two letters placed at the ends of the segment,[6] much the same way as in Aristotle. Euclid used the language of *line* and *surface* instead of numbers and their products. In printed editions of the *Elements* it became quite customary to render the subject more concrete by writing illustrative numerical values alongside the letters. For example, Clavius in 1612 writes (Book VII, Prop. 5, scholium) "A, $\frac{2}{9}$ D $\frac{3}{7}$," and again (Book VIII, Prop. 4), "A, 6.B, 5.C, 4.D, 3." In Robert Simson's translation of Euclid and in others, the order of the English Alphabet is substituted for that of the Greek, thus Α Β Γ Δ Ε Ζ Η Θ, etc., in Euclid are A B C D E F G H, etc., in Simson and others.[7]

[1] Aristotle *Physics* vii. 5.

[2] *Ibid.* viii. 10.

[3] Aristotle *Analytica posteriora* i. 5, p. 74 *a* 17. Reference taken from Gow, *History of Greek Mathematics* (Cambridge, 1884), p. 105, n. 3.

[4] Euclid's *Elements*, Book 7.

[5] Euclid's *Elements*, Book 7, Prop. 3 (ed. J. L. Heiberg), Vol. II (1884), p. 194–98.

[6] Euclid's *Elements*, Book 7, Prop. 1 (ed. Heiberg), Vol. II, p. 188–90.

[7] A. de Morgan in *Companion to the British Almanac*, for 1849, p. 5.

According to Pappus,[1] it was Apollonius of Perga who, like Archimedes, divided numbers into groups or myriads and spoke of double, triple myriads, and so on, and finally of the "κ fold" myriad. This general expression of a myriad of as high an order as we may wish marks a decided advance in notation. Whether it was really due to Apollonius, or whether it was invented by Pappus, for the more elegant explanation of the Apollonian system, cannot now be determined. But Apollonius made use of general letters, in the manner observed in Euclid, as did also Pappus, to an even greater extent.[2] The small Greek letters being used to represent numbers, Pappus employed the Greek capitals to represent general numbers.[3] Thus, as Cantor says, "The possibility presents itself to distinguish as many general magnitudes as there are capital letters."[4]

It is of some interest that Cicero,[5] in his correspondence, used letters for the designation of quantities. We have already seen that Diophantus used Greek letters for marking different powers of the unknown and that he had a special mark $\overset{\circ}{\mu}$ for given numbers. We have seen also a symbol *rû* for known quantities, and *yâ* and other symbols for unknown quantities (Vol. I, § 106).

389. *Middle Ages.*—The Indian practice of using the initial letters of words as abbreviations for quantities was adopted by the Arabs of the West and again by the translators from the Arabic into Latin. As examples, of Latin words we cite *radix, res, census,* for the unknown and its square; the word *dragma* for absolute number.

In Leonardo of Pisa's *Liber abbaci* (1202),[6] the general representation of given numbers by small letters is not uncommon. He and other writers of the Middle Ages follow the practice of Euclid. He uses letters in establishing the correctness of the rules for proving operations by casting out the 9's. The proof begins thus: "To show the foundation of this proof, let *.a.b.* and *.b.g.* be two given numbers which we wish to add, and let *.a.g.* be the number joint from them

[1] Pappus *Collectio,* Book II (ed. Hultsch), Vol. I (1876), p. 2–29. See M. Cantor, *Vorlesungen über Geschichte der Mathematik,* Vol. I (3d ed., 1907), p. 347.

[2] See G. H. F. Nesselmann, *Die Algebra der Griechen* (Berlin, 1842), p. 128–30.

[3] *Pappus* (ed. Hultsch), Vol. I (1876), p. 8.

[4] M. Cantor, *op. cit.,* Vol. I (3d ed., 1907), p. 455.

[5] *Epistolae ad atticum,* Lib. II, epistola 3.

[6] *Scritti di Leonardo Pisano matematico* ... da Bald. Boncompagni (Rome, 1857–62), Vol. I, p. 394, 397, 441. See G. Eneström, *Bibliotheca mathematica* (3d ser.); Vol. XIII (1912–13), p. 181.

. . . ."[1] Observe the use of dots to bring into prominence letters occurring in the running text, a practice very common in manuscripts of that time. In another place Leonardo proposes a problem: *a* horses eat *b* oats in *c* days, *d* horses eat *e* oats in *f* days; if the same amount of feed is eaten in the two cases, then the first product *.a.b.c.* is equal to the second *.d.e.f.*, etc.[2] Still more frequent representation of numbers by letters occurs in Jordanus Nemorarius' *Arithmetica*. Jordanus died 1237; the arithmetic was brought out in print in 1496 and 1514 in Paris by Faber Stapulensis. Letters are used instead of special particular numbers.[3] But Jordanus Nemorarius was not able to profit by this generality on account of the fact that he had no signs of operation—no sign of equality, no symbols for subtraction, multiplication, or division. He marked addition by juxtaposition. He represented the results of an operation upon two letters by a new letter.[4] This procedure was adopted to such an extent that the letters became as much an impediment to rapid progress on a train of reasoning as the legs of a centipede are in a marathon race. Letters are used occasionally in the arithmetic (1321) of the Jewish writer Levi ben Gerson[5] of Avignon who, like Nemorarius, has no signs of operation. Gerson uses letters in treating of permutations and combinations. A similar cumbrousness of procedure with letters is found in the printed editions (1483) of the *Algorismus de integris* of the Italian Prosdocimo de' Beldomandi.[6] Less extensive, but more skilful, use of letters is made in the thirteenth century by Meister Gernardus[7] and in the fourteenth century by the Frenchman N. Oresme,[8] who even prefixed numerical coefficients to the letters in passages like ".4.*a*. excedunt. .3.*a*. in sesquitertia."

390. *Renaissance.*—The employment of letters to represent the various powers of unknown quantities by Chuquet in his manuscript *Triparty*, by Pacioli, and by the early algebraists of the sixteenth

[1] *Scritti di Leonardo Pisano*, Vol. I, p. 20, ll. 9–28.

[2] *Op. cit.*, Vol. I, p. 132, 133.

[3] See M. Curtze in *Zeitschrift f. Mathematik und Physik*, Vol. XXXVI (1891), histor.-liter. Abt., p. 1–3.

[4] P. Treutlein in *Abhandlungen zur Geschichte der Mathematik*, Vol. II (1879), p. 132–33.

[5] Levi ben Gerson, *Die Praxis des Rechners* (trans. G. Lange; Frankfurt a. M., 1909); J. Carlebach, *Lewi ben Gerson als Mathematiker* (Berlin, 1910).

[6] M. Cantor, *op. cit.*, Vol. II (2d ed., 1913), p. 206.

[7] G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XIV (1914), p. 99 ff.

[8] N. Oresme, *Algorismus proportionum* (ed. M. Curtze; Berlin, 1868), p. 22.

century has been explained elsewhere. The manuscript algebra of Adam Riese, found in 1855 in the Library at Marienberg, contains some use of letters to represent given general numbers.[1] Parts of his *Coss* were written in 1524, other parts in the interval 1544–59. Grammateus in his *Rechenbuch* of 1518 uses in one place[2] letters as the terms of a proportion: "Wie sich hadt *a* zum *b* also hat sich *c* zum *d*. Auch wie sich hadt *a* z̃ũ *c* also had sich *b* zum *d*." One finds isolated cases indicating the employment of letters for given numbers in other writers, for instance, Chr. Rudolff (1525) who writes $\surd c+\surd d$, Cardan (1570)[3] who explains that $R\!\!\!/\, \frac{a}{b}$ is equal to $\frac{R\!\!\!/\, a}{R\!\!\!/\, b}$, that is, that $\sqrt{\frac{a}{b}}=\frac{\surd a}{\surd b}$. During the close of the fifteenth and early part of the sixteenth century the development of symbols of operation in algebra proceeded rapidly, but quantities supposed to be given were, as a rule, represented by actual numbers; numerical coefficients were employed with few exceptions. A reader who goes over the explanations of quadratic and cubic equations in works of Tartaglia, Cardan, Stifel, is impressed by the fact. As yet literal coefficients, as we write them in $ax^2+bx=c$ and $x^3+ax=b$, were absent from algebra. In consequence there could not be a general treatment of the solution of a cubic. In its place there was given a considerable number of special cases, illustrated by equations having particular numerical coefficients appropriately chosen for each case. Thus, Cardan,[4] on August 4, 1539, discusses the irreducible case of the cubic, not by considering $x^3=ax+b$, when $\left(\frac{a}{3}\right)^3>\left(\frac{b}{2}\right)^2$, but by taking $x^3=9x+10$, where $27>25$.

391. *Vieta in 1591.*—The extremely important step of introducing the systematic use of letters to denote general quantities and general numbers as coefficients in equations is due to the great French algebraist F. Vieta, in his work *In artem analyticam isagoge* (Tours, 1591). He uses capital letters which are primarily representatives of lines and surfaces as they were with the Greek geometricians, rather than pure numbers. Owing to this conception, he stresses the idea of homogeneity of the terms in an equation. However, he does not confine himself to three dimensions; the geometric limitation is aban-

[1] Bruno Berlet, *Adam Riese, sein Leben, seine Rechenbücher und seine Art zu rechnen. Die Cos von Adam Riese* (Leipzig, 1892), p. 35–62.

[2] Grammateus, *Rechenbuch* (1518), Bl. CIII; J. Tropfke, *Geschichte der Elementar-Mathematik*, Vol. II (2d ed., 1921), p. 42.

[3] J. Tropfke, *op. cit.*, Vol. II (2d ed., 1921), p. 42.

[4] M. Cantor, *op. cit.*, Vol. II (?d ed., 1913), p. 489.

doned, and he proceeds as high as ninth powers—*solido-solido-solidum*. The homogeneity is illustrated in expressions like "*A planum* $+$*Z in B*," the A is designated *planum*, a "surface," so that the first term may be of the same dimension as is the second term, Z times B. If a letter B represents geometrically a length, the product of two B's represents geometrically a square, the product of three B's represents a cube.

Vieta uses capital vowels for the designation of unknown quantities, and the consonants for the designation of known quantities. His own words are in translation: "As one needs, in order that one may be aided by a particular device, some unvarying, fixed and clear symbol, the given magnitudes shall be distinguished from the unknown magnitudes, perhaps in this way that one designate the required magnitudes with the letter A or with another vowel E, I, O, U, Y, the given ones with the letters B, G, D or other consonants."[1]

392. *Descartes in 1637.*—A geometric interpretation different from that of Vieta was introduced by René Descartes in his *La géométrie* (1637). If b and c are lengths, then bc is not interpreted as an area, but as a length, satisfying the proportion $bc:b=c:1$. Similarly, $\frac{b}{c}$ is a line satisfying the proportion $\frac{b}{c}:1=b:c$.

With Descartes, if b represents a given number it is always a positive number; a negative number would be marked $-b$. It was J. Hudde[2] who first generalized this procedure and let a letter B stand for a number, positive *or* negative.

393. *Different alphabets.*—While the Greeks, of course, used Greek letters for the representation of magnitudes, the use of Latin letters became common during the Middle Ages.[3] With the development of other scripts, their use in mathematics was sometimes invoked. In 1795 J. G. Prändel expressed himself on the use of Latin type in algebraic language as follows:

"Why Latin and Greek letters are chosen for algebraic calculation, while German letters are neglected, seems, in books composed in

[1] Vieta, *Isagoge* (Tours, 1591), fol. 7.

[2] J. Hudde, *De reductione aequationum* (1657), published at the end of the first volume of F. Van Schooten's second Latin edition of René Descartes' *Géométrie* (Amsterdam, 1659), p. 439. See G. Eneström in *Encyclopédie des scien. math.*, Tom. I, Vol. II (1907), p. 1, n. 2; also *Bibliotheca mathematica* (3d ser.), Vol. IV (1903), p. 208; *The Geometry of Descartes*, by Smith & Latham, Open Court (Chicago, 1925), p. 301.

[3] See, for instance, Gerbert in *Œuvres de Gerbert*, par A. Olleris (Paris, 1867), p. 429–45.

our language, due to the fact that thereby algebraic quantities can be instantaneously distinguished from the intermixed writing. In Latin, French and English works on algebra the want of such a convenience was met partly by the use of capital letters and partly by the use of italicized letters. After our German language received such development that German literature flourishes in other lands fully as well as the Latin, French and English, the proposal to use German letters in Latin or French books on algebra could not be recounted as a singular suggestion."[1]

"The use of Greek letters in algebraic calculation, which has found wide acceptance among recent mathematicians, cannot in itself encumber the operations in the least. But the uncouthness of the Greek language, which is in part revealed in the shape of its alphabetic characters, gives to algebraic expressions a certain mystic appearance."[2]

Charles Babbage[3] at one time suggested the rule that all letters that denote quantity should be printed in italics, but all those which indicate operations should be printed in roman characters.

The detailed use of letters and of subscripts and superscripts of letters will be treated under the separate topics of algebra and geometry.[4]

That even highly trained mathematicians may be attracted or repelled by the kind of symbols used is illustrated by the experience of Weierstrass who followed Sylvester's papers on the theory of algebraic forms until Sylvester began to employ Hebrew characters which caused him to quit reading.[5]

394. *Astronomical signs.*—We insert here a brief reference to astronomical signs; they sometimes occur as mathematical symbols. The twelve zodiacal constellations are divisions of the strip of the celestial sphere, called the "zodiac"; they belong to great antiquity.[6] The symbols representing these constellations are as follows:

[1] Johann Georg Prändel's *Algebra* (München, 1795), p. 4. [2] *Ibid.*, p. 20.

[3] Charles Babbage, art. "Notation," in *Edinburgh Encyclopedia* (Philadelphia, 1832).

[4] Consult Vol. I, §§ 141, 148, 176, 188, 191, 198, 342, 343; Vol. II, 395–401, 443, 444, 561, 565, 681, 732.

[5] E. Lampe in *Naturwissenschaftliche Rundschau*, Vol. XII (1897), p. 361; quoted by R. E. Moritz, *Memorabilia mathematica* (1914), p. 180.

[6] Arthur Berry, *Short History of Astronomy* (New York, 1910), p. 13, 14; W. W. Bryan, *History of Astronomy* (London, 1907), p. 3, 4; R. Wolf, *Geschichte der Astronomie* (München, 1877), p. 188–91; Gustave Schlegel, *Uranographie chinoise*, Vol. I (Leyden, 1875), Book V, "Des zodiaques et des planètes."

♈	Aries, the Ram	♎	Libra, the Balance
♉	Taurus, the Bull	♏	Scorpio, the Scorpion
♊	Gemini, the Twins	♐	Sagitarius, the Archer
♋	Cancer, the Crab	♑	Capricornus, the Goat
♌	Leo, the Lion	♒	Aquarius, the Water-Bearer
♍	Virgo, the Maid	♓	Pisces, the Fishes

The signs for the planets, sun, moon, etc., are as follows:

☉	Sun	♃	Jupiter
☾	Moon	♄	Saturn
⊕	Earth	☊	Ascending node
☿	Mercury	☋	Descending node
♀	Venus	☌	Conjunction
♂	Mars	☍	Opposition

According to Letronne,[1] the signs for the five planets and the sun and moon occur in two manuscripts of the tenth century; these signs, except that for the moon, are not found in antiquity. The early forms of the signs differ somewhat from those given in printed books. The signs for ascending and descending nodes of the moon's orbit occur in a Greek manuscript of the fourteenth century.[2] Some forms bear resemblance to the Hindu-Arabic numerals. Particularly, those for Jupiter and Saturn look like the four and five, respectively. In the twelfth century there were marked variations in the forms of the Hindu-Arabic numerals and also in the forms of the signs for the planets, sun, and moon. It is believed by some[3] that these astronomical signs and numeral signs (being used often by the same persons) mutually influenced each other, with regard to their forms, before the introduction of printing. Hence the resemblances.

Several of the astronomical signs appear as mathematical symbols. Apparently, the sign for Pisces was chosen by L. and T. Digges as their sign of equality, but they added an additional horizontal stroke, as a cross-line. Such strokes were applied by them also to their symbols for powers of the unknown (Vol. I, § 170). The sign for Taurus, placed horizontally, with the open end to the left, was, we believe, the sign of

[1] Letronne, *Revue archéologique* (1st ser.), Vol. III (Paris, 1846), p. 153, 253–63.

[2] P. Tannery, *Mémoires scientifiques*, Vol. IV (1920), p. 356, 359. Tannery gives facsimile reproductions.

[3] G. Horn–D'Arturo, "Numeri Arabici e simboli celesti," *Pubblicazioni dell'Osservatorio astronomico della R. Università di Bologna*, Vol. I (Roma, 1925), p. 187–204.

equality in the 1637 edition of Descartes' *Géométrie* (§ 191). The sign for Aries, placed horizontally, serves with Kästner[1] for "greater than" and "smaller than." The sign for earth is employed extensively in the modern logical exposition of algebra (§ 494). Extensive use of astronomical signs occurs in Leibniz' letters[2] to Jacob Bernoulli; for instance, $\int ♃ \sqrt[e]{\ } ☽ \, dx$, where each astronomical sign stands for a certain analytic expression (§ 560). Kästner employed the signs for Sun, Moon, Mars, Venus, Jupiter, in the marking of equations, in the place of our modern Roman or Hindu-Arabic numerals.[3] Cauchy sometimes let the sign for Taurus stand for certain algebraic expressions.[4]

THE LETTERS π AND e

395. *Early signs for* 3.1415. . . . —John Wallis,[5] in his *Arithmetica infinitorum* (1655), lets the square □ or, in some cases, the Hebrew letter "mem" which closely resembles a square, stand for 4/3,14149 . . . . ; he expresses □ as the ratio of continued products and also, as William Brounker had done before him, in the form of a continued fraction.

Perhaps the earliest use of a single letter to represent the ratio of the length of a circle to its diameter occurs in a work of J. Chr. Sturm,[6] professor at the University of Altdorf in Bavaria, who in 1689 used the letter e in a statement, "si diameter alicuius circuli ponatur a, circumferentiam appellari posse ea (quaecumque enim inter eas fuerit ratio, illius nomen potest designari littera e)." Sturm's letter failed of general adoption.

Before Sturm the ratio of the length of a circle to its diameter was represented in the fractional form by the use of two letters. Thus,

[1] A. G. Kästner, *Anfangsgründe der Arithmetik, Geometrie* . . . . (Göttingen, 1758), p. 89, 385.

[2] C. I. Gerhardt, *Leibnizens Mathematische Schriften*, Vol. III (Halle, 1855), p. 100.

[3] A. G. Kästner, *Anfangsgründe der Analysis endlicher Grössen* (Göttingen, 1760), p. 55, 269, 336, 358, 414, 417, and other places.

[4] A. L. Cauchy, *Comptes rendus*, Vol. XXIV (1847); *Œuvres complètes* (1st ser.), Vol. X, p. 282.

[5] John Wallis, *Arithmetica infinitorum* (Oxford, 1655), p. 175, 179, 182.

[6] J. Christoph Sturm, *Mathesis enucleata* (Nürnberg, 1689), p. 81. This reference is taken from A. Krazer's note in *Euleri opera omnia* (1st ser.), Vol. VIII, p. 134.

William Oughtred[1] designated the ratio (§ 185) by $\frac{\pi}{\delta}$. He does not define π and δ separately, but no doubt π stood for periphery and δ for diameter. The radius he represents by R. We quote from page 66 of the 1652 edition: "Si in circulo sit $7.22::\delta.\pi::113.355$: erit $\delta.\pi::2R.P$: periph. Et $\pi.\delta::\frac{1}{2}P.R$: semidiam. $\delta.\pi::Rq$. Circul. Et $\pi.\delta::\frac{1}{4}Pq$. Circul." Oughtred's notation was adopted by Isaac Barrow[2] and by David Gregory.[3] John Wallis[4] in 1685 represented by π the "periphery" described by the center of gravity in a revolution. In 1698 De Moivre[5] designated the ratio of the length of the circle to the *radius* by $\frac{c}{r}$.

396. *First occurrence of the sign* π.—The modern notation for 3.14159 . . . . was introduced in 1706. It was in that year that William Jones[6] made himself noted, without being aware that he was doing anything noteworthy, through his designation of the ratio of the length of the circle to its diameter by the letter π. He took this step without ostentation. No lengthy introduction prepares the reader for the bringing upon the stage of mathematical history this distinguished visitor from the field of Greek letters. It simply came, unheralded, in the following prosaic statement (p. 263):

"There are various other ways of finding the *Lengths* or *Areas* of particular *Curve Lines*, or *Planes*, which may very much facilitate the Practice; as for instance, in the *Circle*, the Diameter is to the Circumference as 1 to $\overline{\frac{16}{5}-\frac{4}{239}}-\frac{1}{3}\overline{\frac{16}{5^3}-\frac{4}{239^3}}-$, &c.$=3.14159$, &c.$=\pi$. This *series* (among others for the same purpose, and drawn from the same Principle) I received from the Excellent Analyst, and my much esteem'd Friend Mr. *John Machin;* and by means thereof, *Van*

[1] W. Oughtred, *Clavis mathematicae* (1652), p. 66. This symbolism is given in the editions of this book of 1647, 1648, 1652, 1667, 1693, 1694. It is used also in the Appendix to the *Clavis*, on "Archimedis de Sphaera et Cylindro declaratio." This Appendix appeared in the editions of 1652, 1667, 1693.

[2] W. Whewell, *The Mathematical Works of Isaac Barrow* (Cambridge, 1860), p. 380, Lecture XXIV.

[3] David Gregory, *Philosophical Transactions*, Vol. XIX (London, 1697), p. 652 except that he writes $\frac{\pi}{\rho}$, ρ being the radius.

[4] John Wallis, *Treatise of Algebra* (1685), "Additions and Emendations," p. 170.

[5] De Moivre, *Philosophical Transactions*, Vol. XIX (1698), p. 56.

[6] William Jones, *Synopsis palmariorum matheseos* (London, 1706), p. 263.

Ceulen's Number, or that in Art. 64.38. may be Examin'd with all desirable Ease and Dispatch." Then he writes "$d=c\div\pi$" and "$c=d\times\pi$."

This was not the first appearance of the letter π in Jones's book of 1706. But in earlier passages the meanings were different. On page 241 it was used in lettering a geometric figure where it represented a point. On page 243 one finds "Periphery (π)," as previously found in Wallis.

Nor did the appearance of $\pi=3.14159\ldots.$ on the stage attract general attention. Many mathematicians continued in the old way. In 1721 P. Varignon[1] wrote the ratio $\delta.\pi$, using for ratio the dot of Oughtred.

397. *Euler's use of* π.—In 1734 Euler[2] employed p instead of π and g instead of $\frac{\pi}{2}$. In a letter of April 16, 1738, from Stirling to Euler, as well as in Euler's reply, the letter p is used.[3] But in 1736 he[4] designated that ratio by the sign $1:\pi$ and thus either consciously adopted the notation of Jones or independently fell upon it. Euler says: "Si enim est $m=\frac{1}{2}$ terminus respondens inuenitur $\frac{\pi}{2}$ denotante $1:\pi$ rationem diametri ad peripheriam." But the letter is not restricted to this use in his *Mechanica*, and the definition of π is repeated when it is taken for 3.1415. . . . . He represented 3.1415 . . . . again by π in 1737[5] (in a paper printed in 1744), in 1743,[6] in 1746,[7] and in 1748.[8] Euler and Goldbach used $\pi=3.1415\ldots.$ repeatedly in their correspondence in 1739. Johann Bernoulli used in 1739, in his correspondence with Euler, the letter c (*circumferentia*), but in a letter of 1740

[1] Pierre Varignon, *Histoire de l'Académie r. des sciences*, année 1721 (Paris, 1723), *Mémoires*, p. 48.

[2] Euler in "De summis serierum reciprocarum," *Comm. Acad. Petr.*, Vol. VII (1734–35), p. 123 ff. See von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Vol. II (Leipzig, 1903), p. 110.

[3] Charles Tweedie's *James Stirling* (Oxford, 1922), p. 179, 180, 185, 188.

[4] L. Euler, *Mechanica sive motus scientia analytice exposita*, Vol. I (Petrograd, 1736), p. 119, 123; Vol. II, p. 70, 80.

[5] L. Euler in *Comm. Acad. Petr. ad annum 1737*, IX (1744), p. 165. See A. von Braunmühl, *op. cit.*, Vol. II, p. 110. Euler says: "Posito π pro peripheria circuli, cuius diameter est 1, ... "

[6] L. Euler in *Miscellanea Berolinensia*, Vol. VII (1743), p. 10, 91, 136.

[7] L. Euler in *Histoire de l'académie r. des sciences, et de belles lettres*, année 1745 (Berlin, 1746), p. 44.

[8] L. Euler, *op. cit.*, année 1748 (Berlin, 1750), p. 84.

he began to use π. Likewise, Nikolaus Bernoulli employed π in his letters to Euler of 1742.[1] Particularly favorable for wider adoption was the appearance of π for 3.1415 . . . . in Euler's *Introductio in analysin infinitorum* (1748). In most of his later publications, Euler clung to π as his designation of 3.1415. . . . .

398. *Spread of Jones's notation.*—In 1741, $\pi=3.14159$ . . . . is used in Sherwin's *Tables.*[2] Nevertheless, mathematicians in general were slow in following suit. In 1748 Diderot[3] wrote, "Soit le rapport du diametre à la circumférence $=\frac{1}{c}$ ... ." J. A. Segner varied in his practice; in 1751[4] he let π stand for the ratio, but in 1767 he[5] represented 3.14159 . . . . by $\delta:\pi$, as did Oughtred more than a century earlier. Says Segner: "Si ratio diametri ad peripheriam circuli, quam dedimus, vel alia verae satis propinqua, $\delta:\pi$, et sit diameter circuli data d, erit eiusdem circuli peripheria$=\frac{\pi}{\delta}.d$." Again, later, he lets π be "dimidium peripheriae" of the circle.[6] Even more vacillating was Kästner, who in his *Anfangsgründe* of 1758 lets $1:P$ stand for the ratio of diameter to circumference,[7] and π for the circumference. He uses P in this sense in his plane geometry, and the early part of his solid geometry. Then all of a sudden he writes (p. 323) the ratio in the form $1:\pi$ and continues this notation over nine consecutive pages. Further on (p. 353) in his trigonometry he puts $\cos u=\pi$ and $\sin u=p$; he writes, on page 367, $\cos A=\pi$, and on page 389, $\cos AP=\pi$. It cannot be said that in 1758 Kästner had settled upon any one fixed use of the letter π. In 1760 his practice had not changed;[8] he lets π be coefficient of the $(n+1)$th term of an equation; later he puts π equal to the algebraic irrational $\sqrt{a}$,then $\pi=\sqrt[a]{-1}$, then π is an angle APM

[1] See Paul H. von Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du XVIII siècle* (1843). Also F. Rudio, *Archimedes, Huygens, Lambert, Legendre* (Leipzig, 1892), p. 53.

[2] H. Sherwin's *Mathematical Tables* (3d éd.; revised by William Gardiner, London, 1741), p. 44.

[3] Denys Diderot, *Mémoires sur différens sujets de mathematiques* (Paris, 1748), p. 27.

[4] J. A. Segner, *Histoire de l'académie,* année 1751 (Berlin, 1753), p. 271.

[5] J. A. de Segner, *Cursus Mathematici,* Pars I (2d ed.; Halae, 1767), p. 309.

[6] Segner, *op. cit.*, Pars IV (Halae, 1763), p. 3.

[7] A. G. Kästner, *Anfangsgründe der Arithmetik, Geometrie und Trigonometrie* (Göttingen, 1758), p. 267, 268.

[8] A. G. Kästner, *Anfangsgründe der Analysis endlicher Grössen* (Göttingen, 1760), p. 107, 117, 211, 228, 254, 296, 326, 327, 413, 432.

and $\pi=R$, then $\pi=\tan MpH$, then π is a coefficient in the cubic $z^3+\pi z+p=0$. After that $\pi=3.14159\ldots.$, then π is a general exponent of the variable x, and is $=0$ in a particular case and $=3$ in another, then again π is the coefficient of a term in an equation, and an exponent of x. Evidently π was still serving him in the rôle of a general-utility symbol. But in 1771, at last, Kästner[1] regularly reserved $\pi=3.14159\ldots.$.

Nicolas de Beguelin[2] in 1751 adopted the notation $\pi=3.14159\ldots.$ as did also Daniel Bernoulli[3] in 1753, G. W. Krafft[4] in 1753, Daviet de Foncenex[5] in 1759.

Another noted German writer of textbooks of the eighteenth century, W. J. G. Karsten, uses π in the first volume of his *Lehrbegrif*[6] to represent a polygon, and uses no letter for 3.14 . . . . . But in the second volume he is definite: "Wenn man hinführo ein für allemahl die Zahl 3, 1415926 u.s.f. $=\pi$ setzt, ... so ist $p=2r\pi=\pi d$." One finds π for 3.14159 . . . . in publications of C. A. Vandermonde[7] in 1770, and Laplace[8] in 1782. About the middle of the eighteenth century the letter π was used frequently by French mathematicians in mechanics and astronomy for other designations than 3.141 . . . . , but in the latter part of that century 3.141 . . . . came to be generally designated by π. Unusual is the procedure of Wessel,[9] who writes $\pi=360°$, and of L. N. M. Carnot, who, in his *Géométrie de position* (1803), page 138, takes the radius to be unity and *a-fourth* of the length of the circle to be π, so that "sin $(\pi\pm a)=+\cos a$." Another unusual procedure is that of D. Lardner,[10] who lets π be the "approxi-

[1] A. G. Kästner, *Dissertationes mathematicae et physicae* (Altenbvrgi, 1771), p. 41, 66, 67.

[2] Beguelin, *op. cit.*, année 1751 (Berlin, 1753), p. 444.

[3] Daniel Bernoulli, *Histoire de l'académie*, année 1753 (Berlin, 1755), p. 156.

[4] Georg Wolffgang Krafft, *Institutiones Geometriae Sublimioris* (Tübingen, 1753), p. 122.

[5] Daviet de Foncenex in *Miscellanea philosophico-mathematica Taurinensis*, Vol. I (1759), p. 130.

[6] W. J. G. Karsten, *Lehrbegrif der gesamten Mathematik. 1. Theil* (Greifswald, 1767), p. 304, 412.

[7] Vandermonde in *Histoire de l'Académie des Sciences*, année 1770 (Paris, 1773), p. 494.

[8] Laplace in *op. cit.*, année 1782 (Paris, 1785), p. 15.

[9] Caspar Wessel, *Essai sur la représentation analytique de la direction* (Copenhague, 1897), p. 15. This edition is a translation from the Danish (1799).

[10] Dionysius Lardner, *The First Six Books of the Elements of Euclid* (London, 1828), p. 278.

mate ratio of the circumference of a circle to its diameter," but does not state which approximate value it represents. The Italian, Pietro Ferroni,[1] in 1782 wrote the capital letters P for 3.14159 . . and Π for 6.283 . . Perhaps the earliest elementary French schoolbook to contain π in regular use was A. M. Legendre's *Éléments de géométrie* (1794), page 121.

399. *Signs for the base of natural logarithms.*—The need of a symbol to represent the base of the natural system of logarithms presented itself early in the development of the calculus. Leibniz[2] used the letter b in letters to Huygens of October 3/13, 1690, and January 27, 1691. In the latter he considers $t=\int \frac{dv}{1-v^2}$ and writes $b^t=\frac{1+v}{1-v}$, "b estant une grandeur constante, dont le logarithme est 1, et le logarithme de 1 estant 0." A reviewer[3] of G. Cheyne's *Fluxionum methodus inversa* writes in 1703, "$\int dx:x=lx$ et $X^x=a^y$. (seu cum $la=1$)$x\ lx=y$," thus suggesting the letter a.

400. *The letter e.*—The introduction of the letter e to represent the base of the natural system of logarithms is due to L. Euler. According to G. Eneström, it occurs in a manuscript written in 1727 or 1728, but which was not published until 1862.[4] Euler used e again in 1736 in his *Mechanica*,[5] Volume I, page 68, and in other places, as well as in articles of the years[6] 1747 and 1751. Daniel Bernoulli[7] used e in this sense in 1760, J. A. Segner[8] in 1763, Condorcet[9] in 1771, Lambert[10] in

[1] Pietro Ferroni, *Magnitudinum exponentialium . . . . theoria Florence* (1782), p. 228, 252.

[2] C. I. Gerhardt, *Leibnizens Mathematische Schriften*, Vol. II (Berlin, 1850), p. 53, 76.

[3] *Acta eruditorum* (Leipzig, 1703), p. 451.

[4] Euler's art., "Meditatio in experimenta explosione tormentorum nuper instituta," in the *Opera posthuma* (1862), Vol. II, p. 800–804. See G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XIV (1913–14), p. 81.

[5] L. Euler, *Mechanica sive motus scientia analytice exposita* (St. Petersburg, 1736), p. 251, 256; also in *Comm. Acad. Petr.*, Vol. VII (1740) p. 146.

[6] L. Euler in *Histoire de l'Academie r. d. sciences et d. belles lettres de Berlin*, année 1745 (Berlin, 1746), p. 185; année 1751 (Berlin, 1753), p. 270.

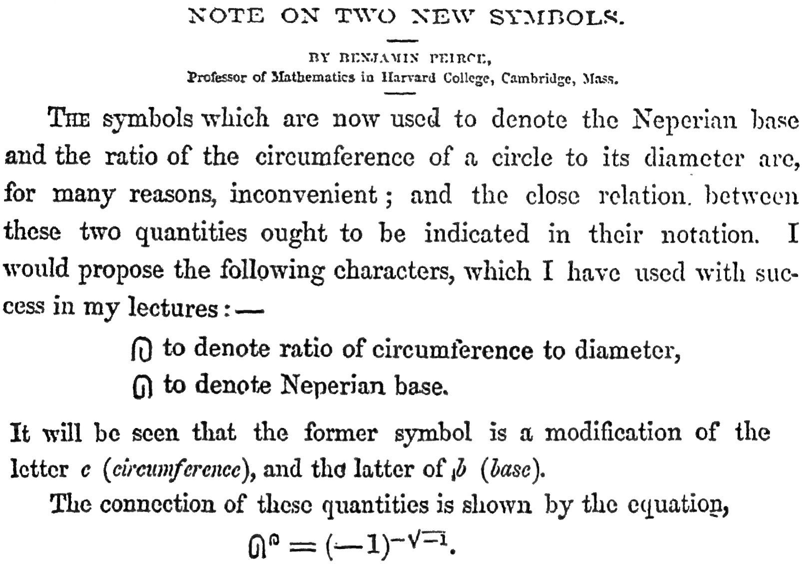
[7] Daniel Bernoulli in *Histoire de l'Académie r. d. sciences*, année 1760 (Paris, 1766), p. 12.

[8] J. A. Segner, *Cursus mathematici*, Paris IV (Halae, 1763), p. 60.

[9] N. C. de Condorcet, *Histoire de l'académie*, année 1771 (Paris, 1774), p. 283.

[10] J. H. Lambert in *Histoire de l'Académie r. d. sciences et d. belles lettres*, année 1764 (Berlin, 1766), p. 188; année 1764 (Berlin, 1766), p. 412.

1764, J. A. Fas[1] in 1775. On the other hand, D'Alembert[2] in 1747 and in 1764 used the letter c for 2.718 . . . . , as did also the astronomer Daniel Melandri[3] of Upsala in 1787. The letter e for 2.718 is found in Abbé Sauri,[4] in E. Bézout,[5] in C. Kramp.[6] In Italy, P. Frisi,[7]



NOTE ON TWO NEW SYMBOLS.

BY BENJAMIN PEIRCE,
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THE symbols which are now used to denote the Neperian base and the ratio of the circumference of a circle to its diameter are, for many reasons, inconvenient; and the close relation between these two quantities ought to be indicated in their notation. I would propose the following characters, which I have used with success in my lectures:—

to denote ratio of circumference to diameter,
to denote Neperian base.

It will be seen that the former symbol is a modification of the letter c (*circumference*), and the latter of b (*base*).

The connection of these quantities is shown by the equation,

$$= (-1)^{-\sqrt{-1}}.$$

FIG. 107.—B. Peirce's signs for 3.141 . . . . and 2.718 . . . .

in 1782, and Pietro Ferroni,[8] in the same year, used C for 2.718 . . . . , but Paoli[9] adopted the e. A. de Morgan[10] in 1842 used the epsilon ϵ for 2.718 . . . . and E for $e^{\sqrt{-1}}$.

[1] J. A. Fas, *Inleiding tot de Kennisse en het gebruyk der Oneindig Kleinen* (Leyden, 1775), p. 71.

[2] D'Alembert in *Histoire de l'académie*, année 1747 (Berlin, 1748), p. 228; année 1764 (Berlin, 1766), p. 412.

[3] Daniel Melandri in *Nova Acta Helvetica physico-mathematica*, Vol. I (Basel, 1787), p. 102.

[4] L'Abbé Sauri, *Cours de mathématiques*, Tome III (Paris, 1774), p. 35.

[5] E. Bézout, *Cours de mathématiques*, Tome I (2d ed.; Paris, 1797), p. 124.

[6] C. Kramp, *Eléments d'arithmétique* (Cologne, 1808), p. 28.

[7] Paulii Frisii, *Operum tomus primus* (Mediolani, 1782), p. 195.

[8] Pietro Ferroni, *Magnitudinum exponentialium logarithmorum et Trigonometriae sublimis theoria* (Florence, 1782), p. 64.

[9] Pietro Paoli, *Elementi d'algebra*, Tomo I (Pisa, 1794), p. 216.

[10] A. de Morgan, "On the Foundations in Algebra," *Transactions of the Cambridge Philos. Society*, Vol. VII (Cambridge, 1842), p. 185.

The use of the letter *c* in place of *e*, found in the writings of a few French and Italian mathematicians, occurs again in the *Analytic Mechanics* of Benjamin Peirce.[1]

401. *B. Peirce's signs for 3.141 .... and 2.718 ....*—An extraordinary innovation in notation for π and e was suggested in 1859 by Benjamin Peirce. He made the statement[2] shown in Figure 107.

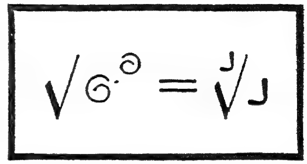


FIG. 108.—From J. M. Peirce's *Tables* (1871)

His sons, Charles Saunders Peirce and James Mills Peirce, used this notation in their articles; the latter placed the symbols shown in Figure 108 on the title-page of his *Three and Four Place Tables* (Boston, 1871). But Peirce's other pupils, Joseph Winlock, Chauncey Wright, and Truman Henry Safford, used the symbol π in the first volume of the *Mathematical Monthly*.

THE EVOLUTION OF THE DOLLAR MARK

402. *Different hypotheses.*—There are few mathematical symbols the origin of which has given rise to more unrestrained speculation and less real scientific study than has our dollar mark, $. About a dozen different theories have been advanced by men of imaginative minds, but not one of these would-be historians permitted himself to be hampered by the underlying facts. These speculators have dwelt with special fondness upon monogrammatic forms, some of which, it must be admitted, maintain considerable antecedent probability. Breathes there an American with soul so dead that he has not been thrilled with patriotic fervor over the "U.S. theory" which ascribes the origin of the $ mark to the superposition of the letters *U* and *S*? This view of its origin is the more pleasing because it makes the symbol a strictly American product, without foreign parentage, apparently as much the result of a conscious effort or an act of invention as is the sewing machine or the cotton gin. If such were the case, surely some traces of the time and place of invention should be traceable; there ought to be the usual rival claimants. As a matter of fact, no one has ever advanced real evidence in the form of old manuscripts, or connected the symbol with a particular place or individual. Nor

[1] Benjamin Peirce, *Analytic Mechanics* (New York, 1855), p. 52.

[2] J. D. Runkle's *Mathematical Monthly*, Vol. I, No. 5 (February, 1859), p. 167, 168, "Note on Two New Symbols."

have our own somewhat extensive researches yielded evidence in support of the "U.S. theory." The theory that the $ is an entwined *U* and *S*, where *U S* may mean "United States" or one "Uncle Sam," was quoted in 1876 from an old newspaper clipping in the *Notes and Queries* (London);[1] it is given in cyclopedic references. In the absence of even a trace of evidence from old manuscripts, this explanation must give way to others which, as we shall find, rest upon a strong basis of fact. Possibly these statements suffice for some minds. However, knowing that traditional theories are dear to the heart of man, an additional coup de grâce will not be superfluous. The earliest high official of the United States government to use the dollar mark was Robert Morris, the great financier of the Revolution. Letters in his own handwriting, as well as those penned by his secretary, which we have seen,[2] give the dollar mark with only one downward stroke, thus, $. To assume that the symbol is made up of the letters *U* and *S* is to assert that Robert Morris and his secretary did not know what the real dollar symbol was; the letter *U* would demand two downward strokes, connected below. As a matter of fact, the "U.S. theory" has seldom been entertained seriously. Perhaps in derision of this fanciful view, another writer declares "surely the stars and stripes is the obvious explanation."[3]

Minds influenced less by patriotic motives than by ecclesiastical and antiquarian predilections have contributed other explanations of our puzzle. Thus the monogrammatic form of *I H S* (often erroneously interpreted as *Jesus, Hominum Salvator*) has been suggested.[4] The combination of *H S* or *I I S*, which were abbreviations used by the Romans for a coin called *sestertius*, have been advocated.[5] We should expect the supporters of these hypotheses to endeavor to establish an unbroken line of descent from symbols used at the time of Nero to the symbols used in the time of Washington. But sober genealogical inquiries of this sort were never made or, if made, they brought disaster to the hypotheses.

A suggestion worthy of serious attention is the Portuguese symbol, *cifrão*, used in designating "thousands," as in 13$786 (Vol. I, § 94). Somehow this symbol is supposed to have received the new meaning of

[1] *Notes and Queries* (5th ser.), Vol. VI (London, 1876), p. 386; Vol. VII, p. 98.

[2] Letter of 1792, in Harper Memorial Library, University of Chicago; Robt. Morris' Private Letter Book, in MSS Div. of Library of Congress.

[3] *Notes and Queries* (5th ser.), Vol. VI, p. 434.

[4] *Standard Dictionary*, art. "Dollar."

[5] M. Townsend, *U.S., an Index, etc.* (Boston, 1890), p. 420.

"dollar" and to have been transferred from its old position in thousands' place to the new position in front of the number affected. The burden of proof that the two transformations actually took place lies with the advocates of this theory. But such a proof has never been attempted in print. The present writer has examined many books and many manuscripts from which support might be expected for such a theory, if true, but nowhere has he found the slightest evidence. The names of monetary units used in Brazil at the beginning of the nineteenth century were *reis, veintein, tuston, pataca, patacon, cruzado,* and none of these was represented by the symbol $.

An interesting hypothesis is advanced by the noted historian, T. F. Medina, of Santiago de Chile. He suggests that perhaps the dollar mark was derived from the stamp of the mint of Potosi in Bolivia. This stamp was the monogrammatic *p* and *s*. Against the validity of this explanation goes the fact that forms of *p* and *s* were used as abbreviations of the *peso* before the time of the establishment of the mint at Potosi.

All the flights of fancy were eclipsed by those who carried the $ back to the "Pillars of Hercules." These pillars were strikingly impressed upon the "pillar dollar," the Spanish silver coin widely used in the Spanish-American colonies of the seventeenth and eighteenth centuries.[1] The "Pillars of Hercules" was the ancient name of the opposite promontories at the Straits of Gibraltar. The Mexican "globe dollar" of Charles III exhibited between the pillars two globes representing the old and new worlds as subject to Spain. A Spanish banner or a scroll around the Pillars of Hercules was claimed to be the origin of the dollar mark.[2] The theory supposes that the mark stamped on the coins was copied into commercial documents. No embarrassments were experienced from the fact that no manuscripts are known which show in writing the imitation of the pillars and scroll. On the contrary, the imaginative historian mounted his Pegasus and pranced into antiquity for revelations still more startling. "The device of the two pillars was stamped upon the coins" of the people who "built Tyre and Carthage"; the Hebrews had "traditions of the pillars of Jachin and Boaz in Solomon's Temple," "still further back in the remote ages we find the earliest known origin of the symbol in connection with the Deity. It was a type of reverence with the first people of the human race who worshipped the sun and the plains of

[1] *Notes and Queries* (5th ser.), Vol. VII (London, Feb. 24, 1877); *New American Cyclopedia,* Vol. VI (1859), art. "Dollar."

[2] M. Townsend, *op. cit.,* p. 420.

central Asia." The author of this romance facetiously remarks, "from thence the descent of the symbol to our own time is obvious."[1] Strange to say, the ingenious author forgot to state that this connection of the dollar mark with ancient deities accounts for the modern phrase, "the almighty dollar."

FIG. 109.—"Pillar dollar" of 1661, showing the "Pillars of Hercules." (From *Century Dictionary*, under "Pillar.")

Most sober-minded thinkers have been inclined to connect the dollar symbol with the figure 8. We have seen four varieties of this theory. The Spanish dollars were, as a rule, equivalent to eight smaller monetary units, universally known in Spain as *reales* or *reals*. The "pillar dollar" shows an 8 between the two pillars. The Spanish dollar was often called a "piece of eight." What guess could be more natural than that the 8 between two pillars suggested the abbreviation 8, which changed into $? So attractive is this explanation that those who advanced it did not consider it worth while to proceed to the prosaic task of finding out whether such symbols were actually employed in financial accounts by merchants of English and Latin America. Other varieties of theorizing claimed a union of P and 8 ("piece of eight")[2] or of R and 8 ("eight *reales*")[3] or of |8| (the vertical

[1] *American Historical Record*, Vol. III (Philadelphia), p. 407–8; *Baltimore American* (June 3, 1874).

[2] M. Townsend, *op. cit.*, p. 420; *Scribner's Magazine*, Vol. XLII (1907), p. 515.

[3] M. Townsend, *op. cit.*, p. 420.

lines being marks of separation)[1] or of 8/.[2] The "*P*8 theory" has been given in Webster's *Unabridged Dictionary,* not in its first edition, but in the editions since the fourth (1859) or fifth (1864). It is claimed that this widely accepted theory rests on manuscript evidence.[3] One writer who examined old tobacco account-books in Virginia reproduces lithographically the fancifully shaped letter *p* used to represent the "piece of eight" in the early years. This part of his article is valuable. But when it comes to the substantiation of the theory that $ is a combination of *P* and 8, and that the $ had a purely local evolution in

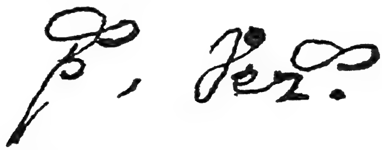


FIG. 110.—Forms that are not dollar symbols

the tobacco districts of Virginia, his facts do not bear out his theory. He quotes only one instance of manuscript evidence, and the reasoning in connection with that involves evident confusion of thought.[4] To us the "*P*8 theory" seemed at one time the most promising working hypothesis, but we were obliged to abandon it, because all evidence pointed in a different direction. We sent inquiries to recent advocates of this theory and to many writers of the present day on early American and Spanish-American history, but failed to get the slightest manuscript evidence in its favor. None of the custodians of manuscript records was able to point out facts in support of this view. We ourselves found some evidence from which a superficial observer might draw wrong inferences. A few manuscripts, particularly one of the year 1696 from Mexico (Oaxaca), now kept in the Ayer Collection of the Newberry Library in Chicago, give abbreviations for the Spanish word *pesos* (the Spanish name for Spanish dollars) which consist of the letter *p* with a mark over it that looks like a horizontal figure 8. This is shown in Figure 110. Is it an 8? Paleographic study goes against this conclusion; the mark signifies *os,* the last two letters in *pesos.* This is evident from several considerations. The fact that in the same manuscript exactly the same symbol occurs in *vez^os^*, the contraction for *vezinos,* or "neighbors," may suffice; an 8 is meaningless here.

[1] *Notes and Queries* (5th ser.), Vol. VII (London), p. 317.

[2] *Scribner's Magazine,* Vol. XLII (1907), p. 515.

[3] *American Historical Record,* Vol. III, p. 271. [4] *Ibid.,* Vol. III, p. 271.

We have now described the various hypotheses.[1] The reader may have been amused at the widely different conclusions reached. One author gives to the \$ "a pedigree as long as chronology itself." Others allow it only about 125 years. One traces it back to the worshipers of the sun in Central Asia, another attributes it to a bookkeeper in a Virginia tobacco district. Nearly every one of the dozen theories seemed so simple to its advocate as to be self-evident.

403. *Evidence in manuscripts and early printed books.*—The history of the dollar mark is difficult to trace. The vast majority of old documents give monetary names written out in full. This is the case also in printed books. Of nine Spanish commercial arithmetics of the seventeenth and eighteenth centuries, five gave no abbreviations whatever for the *peso* (also called *piastre, peso de* 8 *reales,* "piece of eight," "Spanish dollar"). In fact, some did not mention the *peso* at all. The reason for the omission of *peso* is that the part of Spain called Castile had monetary units called *reales, ducados, maravedises,* etc.; the word *peso* was used mainly in Spanish America and those towns of Spain that were in closest touch with the Spanish colonies. After the conquest of Mexico and Peru, early in the sixteenth century, Spanish-American mints, established in the various points in the Spanish possessions, poured forth the Spanish dollar in such profusion that it became a universal coin, reaching before the close of the century even the Philippines and China. In the seventeenth century the Spanish "piece of eight" was known in Virginia, and much was done to promote the influx of Spanish money into that colony. The United States dollar, adopted in 1785, was avowedly modeled on the average weight of the Spanish-dollar coins in circulation. Thomas Jefferson speaks of the dollar as "a known coin, and most familiar of all to the minds of the people."[2] No United States dollars were actually coined before the year 1794.[3] We proceed to unfold our data and to show the evolution of the dollar mark by stages so easy and natural that the conclusion is irresistible. There are no important "missing links." To enable the critical reader to verify our data, we give the sources of our evidence. No man's *ipse dixit* is a law in the world of scientific research.

We begin with information extracted from early Spanish printed books, consisting of abbreviations used for *peso* or *pesos.*

[1] Other possible lines of research on the origin of \$ were suggested by Professor D. E. Smith in his *Rara Arithmetica* (1908), p. 470, 471, 491.

[2] D. K. Watson, *History of American Coinage* (1899), p. 15.

[3] Gordon, *Congressional Currency,* p. 118.

Ivan Vasquez de Serna[1]......1620......*Pes.*, *pes de* 8 *real*
Francisco Cassany[2]..........1763......*p*, also *ps.*
Benito Bails[3]...............1790......*pe*, seldom *p*
Manuel Antonio Valdes[4]......1808......*ps.*

Here we have the printed abbreviations *Pes.*, *ps*, *pe*, *p*. More interesting and convincing are the abbreviations found in manuscripts which record commercial transactions. We can give only a small part of the number actually seen. In our selection we are not discriminating against symbols which might suggest a conclusion different from our own. As a matter of fact, such discrimination would be difficult to make, for the reason that all the abbreviations for the *peso*, or "piece of eight," or *piastre* that we have examined point unmistakably to only one conclusion. We say this after having seen many hundreds of these symbols in manuscripts, antedating 1800, and written in Mexico, the Philippines, San Felipe de Puerto, New Orleans, and the colonies of the United States. It was a remarkable coincidence that all three names by which the Spanish dollar was best known, namely, the *peso*, *piastre*, and "piece of eight," began with the letter *p* and all three were pluralized by the use of the letter *s*. Hence *p* and *ps* admirably answered as abbreviations of any of these names. The symbols in Figure 111 show that the usual abbreviations was *ps* or *p*, the letter *p* taking sometimes a florescent form and the *s* in *ps* being as a rule raised above the *p*. The *p* and the *s* are often connected, showing that they were written in these instances by one uninterrupted motion of the pen. As seen in Figure 111, the same manuscript sometimes shows widely different shapes. The capital *P* is a rare occurrence. We have seen it used at the beginning of sentences and a few times written in ledgers at the top of columns of figures. In the sixteenth century the *ps* had above it a mark indicating the omission of part of the word, thus, *p̆s*. Sometimes the contraction of the word *pesos* was *pss.* or *pos.* Not infrequently two or more different abbreviations are found in one and the same manuscript. The body of the text may contain the word written out in full, or

[1] Ivan Vasquez de Serna, *Reducciones de oro* (Cadiz, 1620), p. 263 ff. (In the Hispanic Museum, New York City.)

[2] Don Fr. Cassany, *Arithmetica deseada* (Madrid, 1763). (In the Library of Congress.)

[3] Don Benito Bails, *Arismetica* (Madrid, 1790). (In the Library of the American Philosophical Society, Philadelphia.)

[4] Don. M. A. Valdes, *Gazetas de Mexico* (1808). (In the Newberry Library, Chicago.)

	Place of MS.	*Date of MS.*	*Date of MS.*	*Place of MS.*	
1	Spain	abt. 1500	1598	Mexico City	2
3	Mexico (?)	1601	1633	San Felipe de puerto	4
5	Mexico	1644	1649	Mexico City	6
7	Manila	1672	1696	Mexico	8
9	Mexico	1718	1746	Mexico City	10
11	Chietla (Mexico)	1748	1766	Manila	12
13	Mexico	1768	1769	?	14
15	New Orleans	1778	(1778) 1783	New Orleans	16
17	Mexico City	1781	1786	New Orleans	18
19	On the Mississippi	1787	1787	Mexico City	20
21	Philadelphia	1792	1793	"Nouvelle Madrid" (N. O.)	22
23	"Nouvelle Madrid" (N. O.)	1794	1794	"Nouvelle Madrid" (N. O.)	24
25	"Nouvelle Madrid" (N. O.)	1794	1794	"Nouvelle Madrid" (N. O.)	26
27	New Orleans	1796	1796	Philadelphia (?)	28
29	New Orleans	1796	1799	Louisville (?)	30

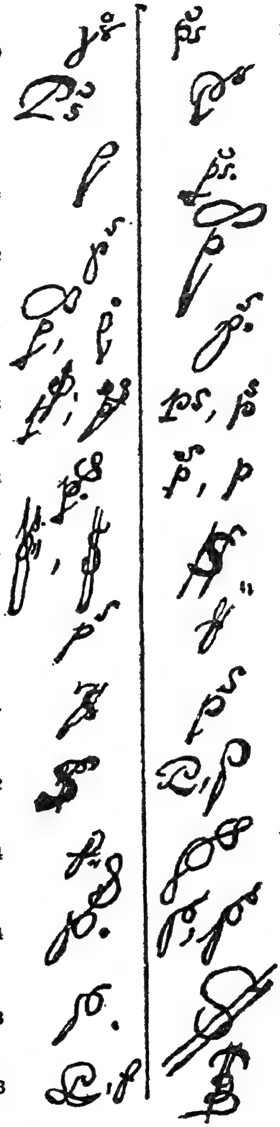


FIG. 111.—Symbols for the Spanish dollar or *peso*, traced from MS letters, contracts, and account-books. No. 1: The historian, Dr. Cayetano Coll y Toste, of Porto Rico, says that this was the written symbol "during the time of Christopher Columbus." Nos. 2, 3, 6, 9, 10, 11, 13, 14, 17, 20 are traced from MSS owned by W. W. Blake, Avenida 16 de Septiembre 13, Mexico City. Nos. 15, 16,

18, 19 are from the Draper Collection in Wis. Hist. Libr., Madison; Nos. 15, 16 in Clark MSS, Vol. XLVIII J, p. 37, 38; Nos. 18, 19 in Clark MSS, Vol. I, p. 136, 143. Nos. 4, 5, 7, 8, 12 are from the Ayer Collection, Newberry Libr., Chicago. No. 21 from letter of Robert Morris to the Hon. Jeremiah Wadsworth, Esq., Hartford, Conn., in Harper Mem. Libr., University of Chicago. Nos. 22, 23, 24, 25, 26, 27, 28, 29, 30 are from MSS in Chicago Hist. Soc. Libr.; No. 22 in the Menard Collection, Vol. LXIV; Nos. 23, 24 in the Menard Collection, Vol. LX, p. 187; Nos. 25, 30 in Autogr. Letters, Vol. LXI; No. 26 in the Menard Collection, Vol. LXII; Nos. 27, 29 in the Menard Collection, Vol. LXIII; No. 28 in Autogr. Letters, Vol. LXXI, p. 76. The "N.O." in the figure, following "Nouvelle Madrid," should be "Mo."

contracted to *pss* or *pos*, while the margin or the head of a column of figures may exhibit *ps* or simply *p*. These were the abbreviations used by the Spanish-Americans from the sixteenth century down to about 1820 or 1830. The transition from the *ps* to our modern dollar mark was not made by the Spaniards; it was made by the English-speaking people who came in contact with the Spaniards. At the time when Mexico achieved its independence (1821), the $ was not yet in vogue there. In a Mexican book of 1834 on statistics[1] both the *ps* and the $ are used. Our $ was introduced into Hawaii by American missionaries in a translation of Warren Colburn's *Mental Arithmetic* in 1835.[2]

The transition from the florescent p^s to our dollar mark is seen in Figure 112. Apparently it is a change introduced unconsciously, in the effort to simplify the complicated motion of the pen called for in the florescent p^s. No manuscript on this point is so interesting and convincing as the two contemporaneous copies, made by the same hand, of a letter written in 1778 by Oliver Pollock, then "commercial agent of the United States at New Orleans." Pollock rendered great service to the United States, being to the west what Robert Morris was to the east. Pollock's letter is addressed to George Roger Clark, who was then heading an expedition for the capture of the Illinois country. Both copies of that letter show the $ in the body of the letter, while in the summary of accounts, at the close, the $ and the florescent p^s are both used. These documents show indeed "the modern dollar mark in the making." In the copy from which our photograph is taken, Figure 112, the 8613 dollars is indicated by the regular $, while in the other copy it is represented by the fancy p^s. Carefully examining the two symbols in our photograph, we see that the p^s is made by one continuous motion of the pen, in this order: Down on the left—up on the right—the loop of the *p*—the *s* above. On the other

[1] J. A. Escudero, *Noticias estadisticas del Estado de Chihuahua* (Mexico, 1834).

[2] Copy in the Newberry Library, Chicago.

hand, the $ symbol is made by two motions: One motion down and up for the *p*, the other motion the curve for the *s*, one symbol being superimposed upon the other.

Mr. Augustus H. Fiske, of Cambridge, Massachusetts, has pointed out to the present writer that the modern dollar mark occurs in a diary of Ezra l'Hommedieu for the year 1776. L'Hommedieu was a native of Southold, Long Island, and a Yale graduate. He was a member of the New York Provincial Assembly, which, on July 10, 1776, styled itself the Convention of the Representatives of the State

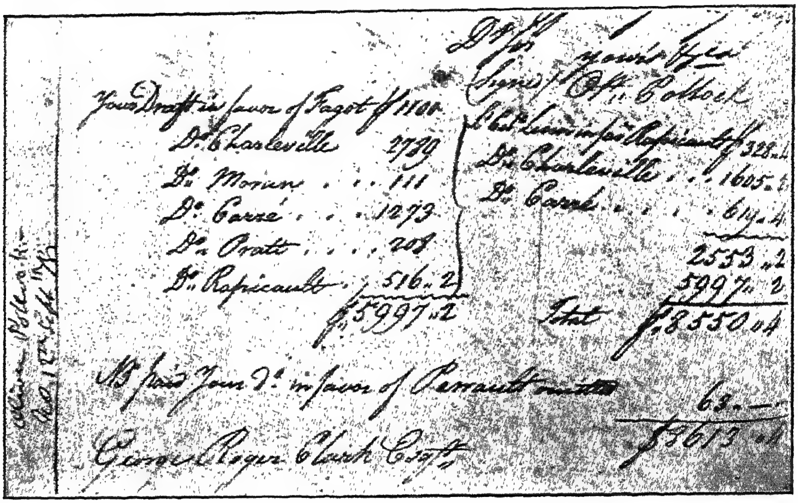


FIG. 112.—The modern dollar mark in the making. (From copy of letter by Oliver Pollock at New Orleans to George Roger Clark, 1778, Wis. Hist. Libr., Madison, Draper Collection, Vol. XXXVIII J, p. 37.)

of New York. The first date in the diary is June 10, 1776; the last is December 5, 1776. Before August 21, 1776, most sums of money are expressed in pounds and shillings. When dollars are mentioned, the word "dollar" is written out in full. On August 21 occurs the first dollar symbol (see tracing 1 in Fig. 113). Under date of August 28 the treasurer is to advance $10 for removing military stores from New York (tracing 2). On October 2 a loan of $100,000 is obtained from the Continental Congress (tracing 3); on October 3 and 4 the same sum is referred to in a similar way (tracings 4 and 5). On October 4 the treasurer is to pay $6412⅔ bounty money to the rangers (tracing 6). The $ signs now appear more frequently. Their shapes are shown in the remaining tracings. We see in this diary the gradual substitu-

tion of the conventional sign $ for the spelled word. The first eleven tracings have the *S* crossed by only one line; the last three have the double lines.

The origin of the dollar mark is simplicity itself. It is an evolution from p^s. When the *p* was made by one long stroke only, as in Figure 111, Nos. 12, 14, 17, 20, then the mark took the form $, as used by Robert Morris (Fig. 111, No. 21). Before 1800 the regular mark $ was seldom used. In all our researches we have encountered it in eighteenth-century manuscripts not more than thirty or forty times. None

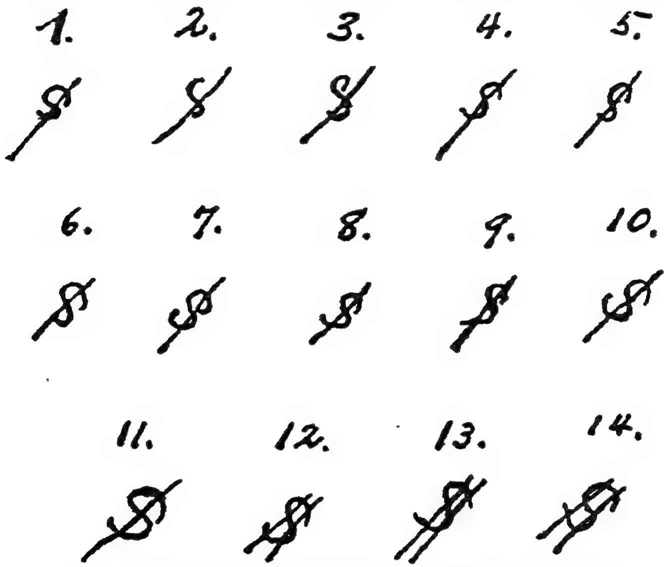


FIG. 113.—Dollar marks in L'Hommedieu's diary, 1776

of these antedates L'Hommedieu's diary of 1776. But the dollar money was then very familiar. In 1778 theater prices in printed advertisements in Philadelphia ran, "Box, one dollar." An original manuscript document of 1780 gives thirty-four signatures of subscribers, headed by the signature of George Washington. The subscribers agree to pay the sum annexed to their respective names, "in the promotion of support of a dancing assembly to be held in Morristown this present winter. The sums are given in dollars, but not one of the signers used the $ symbol; they wrote "Dollars," or "Doll," or "D."[1]

[1] *American Historical and Literary Curiosities* (Philadelphia, 1861), Plates 52, 22.

It is interesting to observe that Spanish-Americans placed the *ps* after the numerals, thus 65*ps*, while the English colonists, being accustomed to write £ before the number of pounds, usually wrote the $ to the left of the numerals, thus $65. It follows after the numerals in some letters written by Joseph Montfort Street, at Prairie du Chien in Wisconsin and Rock Island, Illinois, in 1832 and 1836.[1] The dollar mark $ is usually written after the number, as in 85$, in many Latin-American arithmetics; for instance, in those of Gabriel Izquierdo,[2] Florentino Garcia,[3] Luis Monsante,[4] Agustin de La-Rosa Toro,[5] and Maximo Vazquez.[6] In the Argentine Republic the $ is still frequently written to the right of the numerals, like this, 65$.

404. *Modern dollar mark in print.*—It has been said by various writers, including myself, that the first appearance of the dollar mark in print is in Chauncey Lee's *American Accomptant*, printed at Lansinburgh, New York, in 1797. The statement is inaccurate. Lee's sign for "dimes" more nearly resembles our dollar mark than does his sign for "dollars." We may premise that Lee assumes in his arithmetic the attitude of a reformer. Impressed by the importance of decimal fractions, he declares that "vulgar fractions are a very unimportant, if not useless part of Arithmetic,"[7] and in his Introduction he proceeds to propose a decimal system of weights and measures. In his table "Of Federal Money" (p. 56) he introduces, without comment, new signs for mills, cents, dimes, dollars, and eagles, as shown in Figures

[1] See correspondence of Joseph Montfort Street in the Iowa State Historical Department, in one volume. On p. 31 is a letter from Street to the Hon. Lewis Cass, secretary of the Department of War, Oct. 4, 1832, in which occurs "800 $"; on p. 57 is a letter from Street to David Lowry, Nov., 1836, in which one finds "27000 $," but also "$ 300,000."

[2] Gabriel Izquierdo, *Tratado de aritmética* (Santiago, 1859), p. 219 ff.

[3] Florentino Garcia, *El Aritmético Argentino*, Quinta edicion (Buenos Aires, 1871), p. 24 ff. It is noteworthy that the dollar mark in its modern form occurs in the first edition of this text, 1833. The author, Garcia y Coates, probably followed English texts, for the first edition contains the sign ÷ for division; the author was teaching in the "Academia española e inglesa" in Buenos Aires, and the general arrangement of the original text reminds one of arithmetics in the English language.

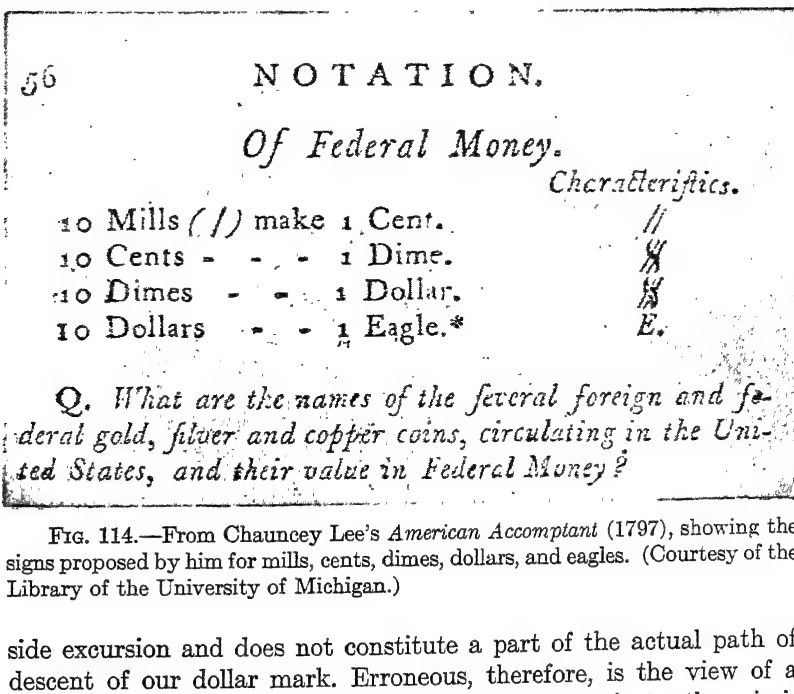
[4] Luis Monsante, *Lecciones de aritmetica demostrada*, Sétima edicion (Lima, 1872), p. 119 ff.

[5] Agustin de La-Rosa Toro, *Aritmetica Teorico-Practica*, Tercera edition (Lima, 1872), p. 126.

[6] *Aritmetica practica* ... por M. Maximo Vazquez ... Séptima edicion (Lima, 1875), p. 113.

[7] Ch. Lee, *American Accomptant* (1797), p. xiv.

114 and 115. It will be seen that his sign for "dollars" involves four strokes, the "dollar" being the fourth denomination of units. Two of the four strokes are curved, and they regularly inclose a space. They do not suggest the letter *S* in p^s from which our dollar mark descended. The probability is that Lee was familiar with the dollar mark as it was found in handwritten business documents of his day, and that he modified it in the manner shown in his book, in order to arrive at a unified plan for constructing signs for mills, cents, dimes, and dollars. It appears, therefore, that Lee's publication marks a



56 NOTATION.

Of Federal Money.

		Characteriſtics.
10 Mills (/) make	1 Cent.	//
10 Cents - - -	1 Dime.	
10 Dimes - -	1 Dollar.	
10 Dollars - -	1 Eagle.*	*E.*

Q. *What are the names of the ſeveral foreign and federal gold, ſilver and copper coins, circulating in the United States, and their value in Federal Money?*

FIG. 114.—From Chauncey Lee's *American Accomptant* (1797), showing the signs proposed by him for mills, cents, dimes, dollars, and eagles. (Courtesy of the Library of the University of Michigan.)

side excursion and does not constitute a part of the actual path of descent of our dollar mark. Erroneous, therefore, is the view of a writer[1] claiming that Lee's symbol for the dollar constitutes the primitive and true origin of the dollar mark.

After 1800 the symbol began to be used, in print, and also more frequently in writing. On September 29, 1802, William A. Washington wrote a letter on the disposal of part of the land above the Potomac belonging to the estate of George Washington. In this letter there is mention of "$20," "$30," and "$40" per acre.[2] In fact, the ledger

[1] See *Bankers Magazine*, Vol. LXII (1908), p. 857.

[2] Letter in Harper Memorial Library, University of Chicago.

kept by George Washington himself, now preserved in the Omaha Public Library, contains the $ frequently. The earliest date in the ledger is January 1, 1799.

The dollar mark occurs a few times in Daniel Adams' *Scholar's Arithmetic; or Federal Accountant,* Keene, New Hampshire (4th ed., 1807) (p. 87, 88). The more common designation in this text is "Dolls." or "D." Adams gives also "D.," "d.," "c.," "m." for dollars, dimes, cents, mills, respectively. With him, the dollar mark has the modern form except that the two strokes are not vertical on the page, but slanting like a solidus. The same form is found in an anonymous publication, "*The Columbian Arithmetician.* By an American" (Haverhill, Mass., 1811).

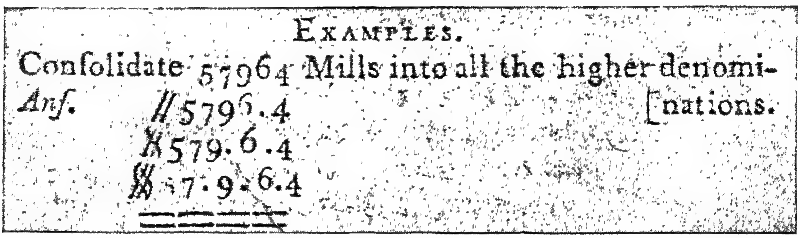

EXAMPLES.
Confolidate 57964 Mills into all the higher denomi-[nations.
Anf. 5796.4
579.6.4
57.9.6.4

FIG. 115.—From Chauncey Lee's *American Accomptant,* p. 142

In newspapers the dollar mark rarely occurs during the first decennium of the nineteenth century. In the *Boston Patriot* of September 1, 1810, one finds "$1 12," (the cents being separated from the dollars by a blank space), but we have failed to find the mark in the numbers of the *Columbian Centinel,* published in Boston, for the period from August to December, 1801, and for August 12, 1809.

Samuel Webber's *Arithmetic* (Cambridge, Mass., 1812) contains the dollar mark but not before page 125. In the first treatment of "Federal Money," page 95, the abbreviations "E.," "D.," "*d.,*" "*c.,*" "*m.*" are used; in the second treatment, page 125, our dollar mark is introduced. The shape of the mark in this book is peculiar. The *S* is unusually broad and is heavy; the two slanting parallel lines are faint, and are close to each other, so short as hardly to project beyond the curves of the *S*, either above or below. Most of the business problems deal with English money.

In Jacob Willetts' *Scholars' Arithmetic* (2d ed., Poughkeepsie, 1817), the dollar mark appears in its modern form with the two strokes straight up and down. Willetts uses also the abbreviations "Dol." and "D."

405. *Conclusion.*—It has been established that the \$ is the lineal descendant of the Spanish abbreviation p^s for *pesos*, that the change from the florescent p^s to \$ was made about 1775 by English-Americans who came into business relations with Spanish-Americans, and that the earliest printed \$ dates back to the opening of the nineteenth century.

SIGNS IN THE THEORY OF NUMBERS

406. In the article on the theory of numbers in the *Encyclopédie des sciences mathématiques*, Tome I, Volume III (1906), page 3, which was originally written by P. Bachmann and later revised by E. Maillet, the great multiplicity and duplication of notations is deplored in the following statement: "Il n'y a malheureusement pas d'entente au sujet des notations relatives aux fonctions arithmétiques qui interviennent dans la théorie des nombres." In the present article we cannot do more than enumerate what seem to be the more important symbols introduced; the preparation of an exhaustive list would seem very onerous and also of comparatively little additional value.

407. *Divisors of numbers; residues.*—The notation $\int n$ for the sum of the divisors of n was introduced by Euler;[1] when n is prime and k an integer, he wrote $\int n^k = 1+n+n^2+ \ldots\ldots +n^k = \frac{n^{k+1}-1}{n-1}$. Barlow[2] employed the sign ⤙ to save repetition of the words "divisible by," and the sign ⇋ to express "of the form of." Cunningham[3] lets "$\sigma(N)$ denote the sum of the sub-factors of N (including 1, but excluding N). It was found that, with most numbers, $\sigma^n N = 1$, when the operation (σ) is repeated often enough." Here $\sigma^2(n)$ means $\sigma\{\sigma(n)\}$. Dickson writes $s(n)$ in place of Cunningham's $\sigma(n)$; Dickson[4] lets $\sigma(n)$ represent the sum of the divisors (including 1 and n) of n; he lets also $\sigma_k(n)$ represent the sum of the kth powers of the divisors of n.

The "symbol of Legendre" is $\left(\frac{N}{c}\right)$; it represents the residue, $+1$

[1] L. Euler, "De numeris amicabilibus," *Opuscula varii argumenti*, Vol. II (Berlin, 1750), p. 23–107; *Commentationes arithmeticae*, Vol. I (Petrograd, 1849), p. 102, 103; *Opera omnia* (1st ser.), Vol. VI, p. 21. See L. E. Dickson, *History of the Theory of Numbers*, Vol. I (1919), p. 42.

[2] Peter Barlow, "Theory of Numbers" in *Encyclopaedia Metropolitana, Pure Sciences*, Vol. I (1845), p. 648.

[3] Allan Cunningham, *Proc. London Math. Soc.*, Vol. XXXV (1902–3), p. 40; L. E. Dickson, *op. cit.*, Vol. I (1919), p. 48.

[4] L. E. Dickson, *op. cit.*, Vol. I, p. 53.

or -1, when $N^{\frac{c-1}{2}}$ is divided by c. In Legendre's words: "Comme les quantités analogues à $N^{\frac{c-1}{2}}$ se recontreront fréquemment dans le cours de nos recherches, nous emploierons la caractère abrégé $\left(\frac{N}{c}\right)$ pour exprimer le reste que donne $N^{\frac{c-1}{2}}$ divisé par c; reste qui, suivant ce qu'on vient de voir, ne peut être que $+1$ ou -1."[1]

Legendre's notation was extended by Jacobi.[2] If $p=ff'f'' \ldots .$, where $f, f', f'' \ldots .$ are uneven prime numbers, then Jacobi defines the symbol $\left(\frac{x}{p}\right)$ by the equation $\left(\frac{x}{p}\right)=\left(\frac{x}{f}\right)\left(\frac{x}{f'}\right)\left(\frac{x}{f''}\right) \ldots . = \pm 1$. In Jacobi's words: "Ist nemlich, um diese Verallgemeinerung für die quadratischen Reste anzudeuten, p irgend eine ungerade Zahl$=ff'f'' \ldots .$, wo $f, f', f'' \ldots .$ gleiche oder verschiedene Primzahlen bedeuten, so dehne ich die schöne Legendre'sche Bezeichnung auf zusammengesetzte Zahlen p in der Art aus, dass ich mit $\left(\frac{x}{p}\right)$, wenn x zu p Primzahl ist, das Product $\left(\frac{x}{f}\right)\left(\frac{x}{f'}\right)\left(\frac{x}{f''}\right) \ldots .$ bezeichne." Legendre's and Jacobi's symbols have been written[3] also $(N|c)$ and $(x|p)$. A sign similar to that of Legendre was introduced by Dirichlet[4] in connection with biquadratic residues; according as $k^{\frac{c-1}{4}} \equiv +1$ or -1 (mod. c), he wrote $\left(\frac{k}{c}\right)_4 = +1$ or -1, respectively. Dickson[5] writes $(k/c)_4$.

Another symbol analogous to that of Legendre was introduced by Dirichlet[6] in the treatment of complex numbers; he designated by $\left[\frac{k}{m}\right]$ the number $+1$, or -1, according as k is, or is not, quadratic residue of m, such that one has $k^{\frac{1}{2}(p-1)} \equiv \left[\frac{k}{m}\right]$ (mod. m), k and m being

[1] A. M. Legendre, *Essai sur la théorie des nombres* (Paris, 1798), p. 186.

[2] C. G. J. Jacobi, *Crelle's Journal*, Vol. XXX (1846), p. 172; *Werke*, Vol. VI (Berlin, 1891), p. 262. Reprinted from *Monatsberichten der Königl. Akademie d. Wissenschaften zu Berlin vom Jahr 1837*. See L. E. Dickson, *op. cit.*, Vol. III, p. 84.

[3] G. B. Mathews, *Theory of Numbers* (Cambridge, 1892), p. 33, 42.

[4] G. L. Dirichlet, *Abhand. d. K. Preussisch. Akad. d. Wissensch.* (1833), p. 101–21; *Werke*, Vol. I (1889), p. 230.

[5] L. E. Dickson, *op. cit.*, Vol. II, p. 370.

[6] G. L. Dirichlet, *Crelle's Journal*, Vol. XXIV (1842), p. 307; *Werke*, Vol. I (1889), p. 551.

complex integers and p the norm of m. Eisenstein[1] employed the sign $\left[\frac{n}{m}\right]$ in biquadratic residues of complex numbers to represent the complex unit to which the power $n^{\frac{1}{4}(p-1)}$ (mod. m) is congruent, where m is any primary prime number, n a primary two-term prime number different from m, p the norm of m.

Jacobi[2] lets $A^{(x)}$ be the excess of the number of divisors of the form $4m+1$, of x, over the number of divisors of the form $4m+3$, of x. His statement is "Sit $B^{(x)}$ numerus factorum ipsius x, qui formam $4m+1$ habent, $C^{(x)}$ numerus factorum, qui formam $4m+3$ habent, facile patet, fore $A^{(x)}=B^{(x)}-C^{(x)}$." Glaisher[3] represents this excess, for a number n, by $E(n)$. Dickson[4] lets $E_r(n)$ or $E'_r(n)$ be the excess of the sum of the rth powers of those divisors of n which (whose complementary divisors) are of the form $4m+1$ over the sum of the rth powers of those divisors which (whose complementary divisors) are of the form $4m+3$. He also lets $\Delta'_r(n)$ be the sum of the rth powers of those divisors of n whose complementary divisors are odd. Bachmann[5] lets $t(n)$, and Landau[6] lets $T(n)$, be the number of divisors of the positive integer n; Landau writes $\mathcal{T}(n)=\Sigma T(n)$, $C=$Euler's constant $0.57721\ldots\ldots$ and $R(x)=\mathcal{T}(x)-x\log x-(2C-1)x$, $x>0$. Dickson lets the sign $T(n)=\mathcal{T}(1)+\mathcal{T}(2)+\ldots\ldots+\mathcal{T}(n)$, where $\mathcal{T}(n)$ is the number of divisors[7] of n. Bachmann[8] designates by $\tilde{\omega}(n)$ the number of distinct prime divisors of n. Landau and Dickson[9] designated by $Og(T)$ a function whose quotient by $g(T)$ remains numerically less than a fixed finite value for all sufficiently large values of T. Before them Bachmann[10] used the form $O(n)$.

The constant C, referred to above as "Euler's constant," was in-

[1] G. Eisenstein, *Crelle's Journal*, Vol. XXX (1846), p. 192.

[2] C. G. J. Jacobi, *Werke*, Vol. I (1881), p. 162, 163; Dickson, *op. cit.*, Vol. I, p. 281.

[3] J. W. L. Glaisher, *Proc. London Math. Soc.*, Vol. XV (1883), p. 104–12.

[4] L. E. Dickson, *op. cit.*, Vol. I, p. 296.

[5] P. Bachmann, *Encyklopädie d. math. Wissensch.*, Vol. I (Leipzig, 1898–1904), p. 648.

[6] Edmund Landau, *Nachrichten von der K. Gesellschaft d. Wissensch. zu Göttingen, Math.-Phys. Klasse* (1920), p. 13.

[7] L. E. Dickson, *op. cit.*, Vol. I, p. 279.

[8] P. Bachmann, *op. cit.*, p. 648.

[9] L. E. Dickson, *op. cit.*, Vol. I, p. 305; Landau, *Handbuch der . . . . Primzahlen*, Vol. I (1909), p. 31, 59.

[10] P. Bachmann, *Analytische Zahlentheorie* (Leipzig, 1894), p. 401.

troduced by Euler[1] who represented it by the letter C. We mention it here, even though strictly it is not part of the theory of numbers. Mertens[2] designated it by a German capital letter $\mathfrak{C}$. It is known also as "Mascheroni's constant."[3] Mascheroni[4] in 1790 designated it by the letter A. This designation has been retained by Ernst Pascal.[5] Gauss[6] wrote $\Psi_0=-0{,}5772156$. W. Shanks[7] adopted the designation "E. or Eul. constant." The letter E for "Eulerian constant" was adopted by Glaisher[8] in 1871 and by Adams[9] in 1878. Unfortunately, E is in danger of being confused with E_{2m}, sometimes used to designate Eulerian numbers.

In formulas for the number of different classes of quadratic forms of negative determinant, Kronecker[10] introduced the following notation which has been adopted by some later writers: n is any positive integer; m is any positive odd integer; r is any positive integer of the form $8k-1$; s is any positive integer of the form $8k+1$; $G(n)$ is the number of non-equivalent classes of quadratic forms of determinant $-n$; $F(n)$ is the number of classes of determinant $-n$, in which at least one of the two outer coefficients is odd; $X(n)$ is the sum of all odd divisors of n; $\Phi(n)$ is the sum of all divisors of n; $\Psi(n)$ is the sum of the divisors of n which are $>\sqrt{n}$, minus the sum of those which are $<\sqrt{n}$; $\Phi'(n)$ is the sum of the divisors of n of the form $8k\pm1$, minus the sum of the divisors of the form $8k\pm3$; $\Psi'(n)$ is the sum both of the divisors of the form $8k\pm1$ which are $>\sqrt{n}$ and of the divisors of the form $8k\pm3$ which are $<\sqrt{n}$, minus the sum of both the divisors of the form $8k\pm1$ which are $<\sqrt{n}$ and of the divisors of the form $8k\pm3$ which are $>\sqrt{n}$; $\phi(n)$ is the number of divisors of n of the form $4k+1$, minus the number of those of the form $4k-1$; $\psi(n)$ is the number of divisors of n of the form $3k+1$, minus the number of those of the form $3k-1$; $\phi'(n)$ is half the number of solutions in integers of $n=x^2+64y^2$;

[1] L. Euler, *Commentarii academiae Petropolitanae ad annum 1736*, Tom. VIII, p. 14–16; M. Cantor, *op. cit.*, Vol. III (2d ed.), p. 665; Vol. IV (1908), p. 277.

[2] F. Mertens, *Crelle's Journal*, Vol. LXXVII (1874), p. 290.

[3] Louis Saalschütz, *Bernoullische Zahlen* (Berlin, 1893), p. 193.

[4] L. Mascheroni, *Adnotationes ad calculum integralem Euleri* (Pavia, 1790–92), Vol. I, p. 11, 60. See also *Euleri Opera omnia* (1st ser.), Vol. XII, p. 431.

[5] Ernst Pascal, *Repertorium d. höheren Mathematik*, Vol. I (1900), p. 477.

[6] C. F. Gauss, *Werke*, Vol. III (Göttingen, 1866), p. 154.

[7] W. Shanks, *Proc. Royl. Soc. of London*, Vol. XV (1867), p. 431.

[8] J. W. L. Glaisher, *Proc. Roy. Soc. of London*, Vol. XIX (1871), p. 515.

[9] J. C. Adams, *Proc. Roy. Soc. of London*, Vol. XXVII (1878), p. 89.

[10] L. Kronecker in *Crelle's Journal*, Vol. LVII (1860), p. 248.

$\psi'(n)$ is half the number of solutions in integers of $n=x^2+3.64y^2$, in which positive, negative, and zero values of x and y are counted for both equations.

Some of the various new notations employed are indicated by the following quotation from Dickson: "Let $\chi_k(x)$ be the sum of the kth powers of odd divisors of x; $\chi_k''(x)$ that for the odd divisors $>\sqrt{x}$; $X_k''(x)$ the excess of the latter sum over the sum of the kth powers of the odd divisors $<\sqrt{x}$ of x; $\chi_k'''(x)$ the excess of the sum of the kth powers of the divisors $8s\pm1>\sqrt{x}$ of x over the sum of the kth powers of the divisors $8s\pm3<\sqrt{x}$ of x."[1]

Lerch[2] of Freiburg in Switzerland wrote in 1905, $q(a)=(a^{p-1}-1)/p$, where p is an odd prime, and a any integer not a multiple of p. Similarly, Dickson[3] lets the sign q_u stand for the quotient $(u^{p-1}-1)/p$, or for $(u^{\phi(p)}-1)/p$.

Möbius[4] defined the function b_m to be zero if m is divisible by a square >1, but to be $(-1)^k$ if m is a product of k distinct primes >1, while $b_1=1$. Mertens[5] writes μn, Dickson,[6] $\mu(n)$, for the b_m of Möbius. This function is sometimes named after Mertens.

Dirichlet[7] used the sign $\left[\frac{n}{s}\right]$, when n and s are integers and $s\leqq n$, to designate the largest integer contained in $\frac{n}{s}$. Mertens[8] and later authors wrote $[x]$ for the largest integer $\leqq x$. Stolz and Gmeiner[9] write $\left[\frac{a}{b}\right]$ or $[a:b]$. Dirichlet[10] denoted by $N(a+bi)$ the norm a^2+b^2 of the complex number $a+bi$, a symbolism used by H. J. S. Smith[11] and others.

[1] L. E. Dickson, *op. cit.*, Vol. I, p. 305.

[2] M. Lerch, *Mathematische Annalen*, Vol. LX (1905), p. 471.

[3] L. E. Dickson, *op. cit.*, Vol. I, p. 105, 109; see also Vol. II, p. 768.

[4] A. F. Möbius in *Crelle's Journal*, Vol. IX (1832), p. 111; Möbius, *Werke*, Vol. IV, p. 598.

[5] F. Mertens, *Crelle's Journal*, Vol. LXXVII (1874), p. 289.

[6] L. E. Dickson, *op. cit.*, Vol. I, p. 441.

[7] G. L. Dirichlet, *Abhand. d. K. Preussisch. Akad. d. Wissensch. von 1849*, p. 69–83; *Werke*, Vol. II (1897), p. 52.

[8] F. Mertens, *op. cit.*, Vol. LXXVII (1874), p. 290.

[9] Stolz und J. A. Gmeiner, *Theoretische Arithmetik*, Vol. I (2d ed.; Leipzig, 1911), p. 29.

[10] G. L. Dirichlet, *Crelle's Journal*, Vol. XXIV (1842), p. 295.

[11] H. J. S. Smith, "Report on the Theory of Numbers," *Report British Association* (London, 1860), p. 254.

The designation by $\dot{n}$ of a multiple of the integer n is indicated in the following quotation from a recent edition of an old text: "Pour exprimer un multiple d'un nombre nous mettrons un point audessus de ce nombre: ainsi ... $\overline{\dot{3}27}$ signifie un multiple de 327 ..., $\overline{a\dot{,}c}$ ... signifie un multiple commun aux deux nombres a, et c."[1]

408. *Congruence of numbers.*—The sign $\equiv$ to express congruence of integral numbers is due to C. F. Gauss (1801). His own words are: "Numerorum congruentiam hoc signo, $\equiv$, in posterum denotabimus, modulum ubi opus erit in clausulis adiungentes, $-16\equiv 9$ (mod. 5), $-7\equiv 15$ (modo 11)."[2] Gauss adds in a footnote: "Hoc signum propter magnam analogiam quae inter aequalitatem atque congruentiam invenitur adoptavimus. Ob eandem caussam ill. Le Gendre in comment. infra saepius laudanda ipsum aequalitatis signum pro congruentia retinuit, quod nos ne ambiguitas oriatur imitari dubitavimus."

The objection which Gauss expressed to Legendre's double use of the sign $=$ is found also in Babbage[3] who holds that Legendre violated the doctrine of *one notation for one thing* by letting $=$ mean: (1) ordinary equality, (2) that the two numbers between which the $=$ is placed will leave the same remainder when each is divided by the same given number. Babbage refers to Peter Barlow as using in his *Theory of Numbers* (London, 1811) the symbol *ff* placed in a horizontal position (see § 407). Thus, says Babbage, Legendre used the symbolism $a^n=-1$, Gauss the symbolism $a^n\equiv -1$ (mod. p), and Barlow the symbolism $a^n \leftrightarrows p$. Babbage argues that "we ought not to multiply the number of mathematical symbols without necessity." A more recent writer expresses appreciation of Gauss's symbol: "The invention of the symbol $\equiv$ by Gauss affords a striking example of the advantage which may be derived from appropriate notation, and marks an epoch in the development of the science of arithmetic."[4]

Among the earliest writers to adopt Gauss's symbol was C. Kramp of Strassbourg; he says: "J'ai adopté de même la signe de *congruence*, proposé par cet auteur, et composé de trois traits parallèles, au lieu de

[1] Claude-Gaspar Bachet, *Problèmes plaisants et délectables* (3d ed., par A. Labosne; Paris, 1874), p. 13.

[2] C. F. Gauss, *Disquisitiones arithmeticae* (Leipzig, 1801), art. 2; *Werke*, Vol. I (Göttingen, 1863), p. 10.

[3] Charles Babbage, art. "Notation" in the *Edinburgh Cyclopaedia* (Philadelphia, 1832).

[4] G. B. Mathews, *Theory of Numbers* (Cambridge, 1892), Part I, sec. 29.

deux. Ce signe m'a paru essentiel pour toute cette partie de l'analyse, qui admet les seules solutions en nombres entiers, tant positifs que négatifs."[1] The sign $\mathrel{\equiv\!\!\!\!\!\!/}$ is sometimes used for "incongruent."[2]

409. *Prime and relatively prime numbers.*—Peano[3] designates a prime by N_p. Euler[4] lets πD stand for the number of positive integers not exceeding D which are relatively prime to D ("denotet character πD multitudinem istam numerorum ipso D minorum, et qui cum eo nullum habeant divisorem communem"). Writing n for D, Euler's function πD was designated $\phi(n)$ by Gauss,[5] and $T(n)$ (totient of n) by Sylvester.[6] Gauss's notation has been widely used; it is found in Dedekind's edition of Dirichlet's *Vorlesungen über Zahlentheorie*[7] and in Wertheim's *Zahlentheorie.*[8] Jordan[9] generalized Euler's πD function, and represented by $[n, k]$ the number of different sets of k (equal or distinct) positive integers $\leqq n$, whose greatest common divisor is prime to n. In place of Jordan's $[n, k]$ Story[10] employed the symbol $\mathcal{T}^k(n)$, some other writers $\phi_k(n)$, and Dickson[11] $J_k(n)$.

Meissel designates by $\psi(n)$ the nth prime number,[12] so that, for instance, $\psi(4)=5$, and by *rev* (*reversio*) the function which is the opposite[13] of ψ, so that "*rev* $\psi(x)=\psi$ *rev* $(x)=x$," and by E *rev* $(m)=\vartheta(m)$ the number of primes in the natural series from 1 to m inclusive,[14]

[1] Cf. Kramp, "Notations," *Élémens d'Arithmétique Universelle* (Cologne, 1808).

[2] See, for instance, L. E. Dickson, *Algebras and Their Arithmetics* (Chicago, 1923), p. 38.

[3] G. Peano, *Formulaire mathématique,* Tom. IV (1903), p. 68.

[4] *Acta Acad. Petrop.*, 4 II (or 8), for the year 1755 (Petrograd, 1780), p. 18; *Commentationes arithmeticae,* Vol. II (Petrograd, 1849), p. 127; L. E. Dickson, *op. cit.*, Vol. I, p. 61, 113.

[5] C. F. Gauss, *Disquisitiones arithmeticae* (Leipzig, 1801), No. 38. See also article by P. Bachmann and E. Maillet in *Encyclopédie des sciences mathématiques,* Tome I, Vol. III (1906), p. 3.

[6] J. J. Sylvester, *Philosophical Magazine* (5th ser.), Vol. XV (1883), p. 254.

[7] P. G. L. Dirichlet, *Vorlesungen über Zahlentheorie,* herausgegeben von R. Dedekind (3d ed.; Braunschweig, 1879), p. 19.

[8] G. Wertheim, *Anfangsgründe der Zahlentheorie* (Braunschweig, 1902), p. 42.

[9] C. Jordan, *Traité des substitutions* (Paris, 1870), p. 95–97.

[10] W. E. Story, *Johns Hopkins University Circulars,* Vol. I (1881), p. 132.

[11] L. E. Dickson, *op. cit.*, Vol. I, p. 147, 148.

[12] E. Meissel, *Crelle's Journal,* Vol. XLVIII, p. 310.

[13] E. Meissel, *ibid.*, p. 307.

[14] E. Meissel, *ibid.*, p. 313.

which Dickson[1] represents by $\theta(m)$. Landau[2] writes $\pi(x)$ for the number of primes $\leqq x$ in the series 1, 2, ... , $[x]$, where $[x]$ is the largest integer $\leqq x$. He also lets $f(x)$ be a simpler function of x, such that $\frac{\pi(x)-f(x)}{\pi(x)} \rightarrow 0$ when $x \rightarrow \infty$. R. D. Carmichael[3] uses $H\{y\}$ to represent the index of the highest power of the prime p dividing y, while Stridsberg[4] uses H_m to denote the index of the highest power of the prime p which divides $m!$

410. *Sums of numbers.*—Leibniz[5] used the long letter $\int$ not only as a sign of integration in his calculus, but also as the sign for the sum of integers. For example, he marked the sum of triangular numbers thus:

$$\text{“}1+3+6+10+\text{etc.}=\int x$$

$$1+4+10+20+\text{etc.}=\int\int x$$

$$1+5+15+25+\text{etc.}=\int^3 x\,.\text{”}$$

This practice has been followed by some elementary writers; but certain modifications were introduced, as, for example, by De la Caille[6] who lets $\int$ stand for the sum of certain numbers, $\int^2$ for the sum of their squares, $\int^3$ for the sum of their cubes, etc. Bachmann[7] lets $\int_m(k)$ stand for the sum of the mth powers of the divisors of k, m being any given odd number. A. Thacker[8] of Cambridge used the notation $\phi(z)=1^n+2^n+\ldots\ldots+z^n$, where z is an integer and n a positive integer. Dickson[9] lets $\phi_k(n)$ be the sum of the kth powers of

[1] L. E. Dickson, *op. cit.*, Vol. I, p. 429.

[2] Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig und Berlin), Vol. I (1909), p. 4.

[3] R. D. Carmichael, *Bull. Amer. Math. Soc.*, Vol. XV (1909), p. 217.

[4] E. Stridsberg, *Arkiv för Matematik, Astr., Fysik.*, Vol. VI (1911), No. 34. See L. E. Dickson, *op. cit.*, Vol. I, p. 264.

[5] G. W. Leibniz, "Historia et origo calculi differentialis" in *Leibnizens Mathematische Schriften* (ed. C. I. Gerhardt), Vol. V (1858), p. 397.

[6] D. de la Caille, *Lectiones elementares mathematicae . . . . in Latinum traductae . . . . a C(arolo) S.(cherffer)e S. J.* (Vienna, 1762), p. 107.

[7] *Encyklopädie d. Math. Wiss.*, Vol. I (1900–1904), p. 638.

[8] A. Thacker, *Crelle's Journal*, Vol. XL (1850), p. 89.

[9] L. E. Dickson, *op. cit.*, Vol. I, p. 140.

the integers $\leqq n$ and prime to n. Sylvester[1] in 1866 wrote $S_{j,i}$ to express the sum of all the products of j distinct numbers chosen from $1, 2, \ldots, i$ numbers. L. E. Dickson[2] writes $S_{n,m}$ for Sylvester's $S_{j,i}$, and also S_n for $1^n+2^n+\ldots+(p-1)^n$. The sign S_n has been variously used for the sum of the nth powers of all the roots of an algebraic equation.[3] The use of Σ for "sum" is given in § 407.

The sign $F(a, N)$ stands for a homogeneous symmetric polynomial, a and N being integers.[4] Dickson[5] introduces the symbol $\stackrel{n}{=}$ in the following quotations: "The separation of two sets of numbers by the symbol $\stackrel{n}{=}$ shall denote that they have the same sum of kth powers for $k=1, \ldots, n$. Chr. Goldbach noted that $\alpha+\beta+\delta$, $\alpha+\gamma+\delta$, $\beta+\gamma+\delta$, $\delta \stackrel{2}{=} \alpha+\delta$, $\beta+\delta$, $\gamma+\delta$, $\alpha+\beta+\gamma+\delta$."

411. *Partition of numbers.*—Euler's "De partitione numerorum"[6] contains no symbolism other than that of algebra. He starts out with the product $(1+x^\alpha z)(1+x^\beta z)(1+x^\gamma z)(1+x^\delta z)(1+x^\epsilon z)$, etc., and indicates its product by

$$1+Pz+Qz^2+Rz^3+Sz^4+, \text{ etc.}$$

and considers the term $Nx^n z^m$ "whose coefficient indicates in what various ways the number n may be the sum of m different terms of the series α, β, γ, δ, ϵ, ζ, etc."

Chrystal[7] uses a quadripartite symbol to denote the number of partitions. "Thus, $P(n|p|q)$ means the number of partitions of n into p parts, the greatest of which is q"; "$P(n|*|\not> q)$ means the number of partitions of n into any number of parts, no one of which is to exceed q; $Pu(n|\not> p|*)$, the number of partitions of n into p or any less number of unequal parts unrestricted in magnitude." "$P(n|*1, 2, 2^2, 2^3, \ldots)$ the number of partitions of n into any number of parts, each part being a number in the series $1, 2, 2^2, 2^3, \ldots$."

C. G. J. Jacobi represents the number of partitions, without repetition, of a number n, by $N(n=1\cdot x_1+2\cdot x_2+3\cdot x_3)$, $x_i \geqq 0$. Sylvester,[8]

[1] J. J. Sylvester, *Giornale di matematiche*, Vol. IV (Napoli, 1866), p. 344.

[2] L. E. Dickson, *op. cit.*, Vol. I, p. 95, 96.

[3] See, for instance, M. W. Drobisch, *Höhere numerische Gleichungen* (Leipzig, 1834), p. 133, 134.

[4] L. E. Dickson, *op. cit.*, Vol. I, p. 84.

[5] L. E. Dickson, *op. cit.*, Vol. II, p. 705.

[6] L. Euler, *Introductio in analysin infinitorum*, Vol. I, Editio Nova (Lugduni 1797), chap. xvi, p. 253.

[7] G. Chrystal, *Algebra*, Part II (Edinburgh, 1889), p. 527, 528.

[8] J. J. Sylvester, *Collected Mathematical Papers*, Vol. II (Cambridge, 1908), "On the Partitions of Numbers," p. 128.

using the terms "denumerant" and "denumeration," writes ƆU and Ɔ(U, V). "ƆU in its explicit form $\frac{n;}{a,b,c;\ \ldots\ l;}$" "Herschel's symbol $r_n, \ldots$ as a linear function of the nth powers of the rth roots of unity, ... will be replaced by $\frac{n;}{r;}$." The coefficient[1] of t^n in the product of the series generated by $\frac{1}{1-t^a}\cdot\frac{1}{1-t^b}\cdot$, etc., is represented as $\frac{n;}{a,b,c,\ \ldots\ l;}$, the meaning of which is then extended and modified by Sylvester.[2] "A system of equations[3] in $x, y, z, \ldots u$, may be denoted by $S(x, y, z, \ldots, u)$ or .... by S alone"; "R_xS is the equation which expresses that the coefficient cluster and primary of S balance about the axis Ox." "If S' is what S becomes when we write in S, $fx+g$, or more generally ϕx, in place of x, we may denote S' symbollically by $\frac{\phi x}{x}S$," where $\frac{\phi x}{x}$ is an operative symbol. "$\frac{n-3;}{1,\ 2,\ 3,\ \ldots,\ r;}$ expresses[4] the r-ary partibility of n when repetitions are allowed; $\frac{n-r\,\frac{r+1;}{2}}{1,\ 2,\ 3\ \ldots\ r;}$ the same when repetitions are excluded."

The sign $p(n)$ is used to express the total number of ways in which a given positive integer n may be broken up into positive integral summands, counting as identical two partitions which are distinguished only by the order of the summands.[5]

When the order of summands is taken into account, then the resulting "compositions" are often marked by $c(n)$. G. H. Hardy and J. E. Littlewood[6] use g_k as a positive integer depending only on k, such that every positive integer is the sum of g_k or fewer positive kth powers; they also use $G(k)$ to represent a number such that every sufficiently large number can be represented as the sum of not more than $G(k)$ positive kth powers. We have $G_2=g_2=4$; $G_3\leqq 8$, $g_3=9$.

412. *Figurative numbers.*—An old symbolism for square numbers is the geometric square □, for triangular numbers the triangle △, and so on. Hence the names "polygonal" numbers and "pyramidal" numbers and the general name "figurate" numbers. L. Euler and C.

[1] *Op. cit.*, p. 132. [2] *Op. cit.*, p. 133. [3] *Op. cit.*, p. 138. [4] *Op. cit.*, p. 155.

[5] A. J. Kempner, *American Mathematical Monthly*, Vol. XXX (1923), p. 357.

[6] G. H. Hardy and J. E. Littlewood, *Nachrichten von der K. Gesellschaft d. Wissensch. zu Göttingen, Math.-Phys. Klasse* (1920), p. 34. See also A. J. Kempner, *op. cit.*, p. 363, 364.

Goldbach used the geometric signs $\square$ and $\triangle$ in connection with number theory in their correspondence,[1] about 1748. Occasionally they write also ▭ to denote the product of two unequal factors. On September 7, 1748, and again later, Goldbach[2] lets $\square$ be an even square and $\boxdot$ an odd square; he lets also $\boxminus$ or $\boxminus$ stand for a square which may be even or odd. Goldbach writes "$4\boxminus+\boxdot+\boxdot+\boxdot=8m+5\pm2$," where, in case $+$ is taken, $\boxminus$ is an odd square, and in case $-$ is taken $\boxminus$ is an even square.

Legendre[3] designates by "pol. x" a polygonal number of the order $m+2$, the side of the polygon being x, so that "pol. $2=m+2$, pol. $3=3m+3$" and pol. x, of the order $m+2$, is $\frac{m}{2}(x^2-x)+x$. For triangular numbers, $m=1$; for square numbers, $m=2$, etc.

In an arithmetical progression with the first term 1 and the common difference $m-2$, the sum of r terms is the rth m-gonal number, designated by $P_r^{(m)}$ in the *Encyclopédie des sciences mathématiques*, Tome I, Volume 1, page 30, but is designated p_m^r by L. E. Dickson,[4] who lets P_r^m stand for the pyramidal number $p'_m+p_m^2+\ldots+p_m^r$. Dickson says further: "We shall often write $\triangle_r$ or $\triangle(r)$ for the rth triangular number $r(r+1)/2$, $\triangle$ or $\triangle'$ for any triangular number, $\square$ for any square, $\boxed{2}$, $\boxed{3}$, or $\boxed{4}$ for a sum of two, three, or four squares. The rth *figurate* number of the order n is the binomial coefficient $f_n^r=\binom{r+n-1}{n}=\frac{(r+n-1)(r+n-2)\ldots r}{1.2\ldots n}$."[5]

413. *Diophantine expressions.*—In the solution of $ax+by=c$ and $ax^2+bx+c=\square$, L. Euler wrote (v, a) for $av+1$, (v, a, b) for $(ab+1)v+b$, etc., while C. F. Gauss[6] employed the notations $A=[a]$, $B=[a, \beta]=\beta A+1$, $C=[a, \beta, \gamma]=\gamma B+A$, $D=[a, \beta, \gamma, \delta]=\delta C+B$, etc.

Gauss[7] employed the notation $\begin{pmatrix} a, & a', & a'' \\ b, & b', & b'' \end{pmatrix}$ for $axx+a'x'x'+a''x''x''$

[1] P. H. Fuss, *Correspondance mathématique et physique ... du XVIIIème siècle* (St. Péterbourg), Vol. I (1843), p. 451–63, 550, 602, 630.

[2] P. H. Fuss, *op. cit.*, Vol. I, p. 476, 492.

[3] A. M. Legendre, *Théorie des nombres* (3d ed.), Vol. II (Paris, 1830), p. 340, 349.

[4] L. E. Dickson, *op. cit.*, Vol. II (1920), p. 1, 2.

[5] L. E. Dickson, *op. cit.*, Vol. II, p. 6, 7.

[6] C. F. Gauss, *Disquisitiones Arithmeticae* (1801), § 27; *Werke*, Vol. I (1863), p. 20. See L. E. Dickson, *op. cit.*, Vol. II, p. 49, 357.

[7] C. F. Gauss, *Disquisitiones Arithmeticae* (1801), arts. 266–85; *Werke*, Vol. I (1863), p. 300. L. E. Dickson, *op. cit.*, Vol. III, p. 206.

$+2bx'x''+2b'xx''+2b''xx'$. G. Eisenstein[1] considered the cubic form

$$f=(a, b, c, d)=ax^3+3bx^2y+3cxy^2+dy^3$$

with integral coefficients and let $A=b^2-ac$, $B=bc-ad$, $C=c^2-bd$. He called $Ax^2+Bxy+Cy^2=F$ the determining form of the cubic form f. The sign (a, b, c) is used by Dickson[2] for the binary quadratic form with integral coefficients $ax^2+2bxy+cy^2$.

Kronecker denoted by $K(D)$ the number of primitive classes of discriminant $D=b^2-4ac$. He put $H(D)=\sum_{h=1}^{\infty}\left(\frac{D}{h}\right)\frac{1}{h}$ and $\left(\frac{D}{h}\right)=\left(\frac{2^g}{D}\right)\left(\frac{D}{h'}\right)$ where $h=2^g h'$, h' uneven, and $\left(\frac{2^g}{D}\right)$ and $\left(\frac{D}{h'}\right)$ are the Jacobi-Legendre signs.[3] Cresse[4] explains the more current notation thus: "$h(D)$ denotes the number of properly primitive, and $h'(D)$ the number of improperly primitive, classes of Gauss forms (a, b, c) of determinant $D=b^2-ac$. Referring to Gauss' forms, $F(D)$, $G(D)$, $E(D)$, though printed in italics, will have the meaning which L. Kronecker assigned to them when printed in roman type. The class-number symbol $H(D)$ is defined as $G(D)-F(D)$. By $K(D)$ or $C(D)$ we denote the number of classes of primitive Kronecker forms of discriminant $D=b^2-4ac$."

Val (ω) is called the "valence of ω" and is a function of ω in a field S; it was used by Dedekind[5] in 1887. In the congruential theory of forms, L. E. Dickson uses E. H. Moore's notation, $GF[p^n]$, to represent[6] the Galois field of order p^n, the letter F being used here as in many other English articles to signify "field."

414. *Number fields*.—The German designation for number field or domain, namely, *Zahlkörper* or simply *Körper*, has given rise to the general notation[7] *Körper* K, used by Dedekind. For special number

[1] G. Eisenstein, *Crelle's Journal*, Vol. XXVII (1844), p. 89; L. E. Dickson, *op. cit.*, Vol. III, p. 253.

[2] L. E. Dickson, *op. cit.*, Vol. III, p. 2.

[3] L. Kronecker, *Sitzungsberichte Akad. d. Wissensch.* (Berlin, 1885), Vol. II, p. 768–80; L. E. Dickson, *op. cit.*, Vol. III, p. 138.

[4] G. H. Cresse in Dickson's *History*, Vol. III, p. 93.

[5] R. Dedekind, *Crelle's Journal*, Vol. LXXXIII (1877), p. 275. See L. E. Dickson, *op. cit.*, Vol. III, p. 125.

[6] L. E. Dickson, *op. cit.*, Vol. I, p. 249; Vol. III, p. 293.

[7] See P. G. L. Dirichlet, *Zahlentheorie*, herausgegeben von R. Dedekind (3d ed.; Braunschweig, 1879), p. 465 n.

fields Dedekind[1] used *Körper R* for the field of rational numbers, *Körper J* for the field of all complex numbers $\omega = x+yi$, where x and y are real and rational. He lets $N(\omega)$ stand for the norm (modulus) of ω and its conjugate ω', *Körper* Ω signifies with him a finite field[2] of degree n, but Ω has been employed in the theory of equations as a more general symbol for field. In 1873, G. Cantor[3] used (Ω) to signify the *Inbegriff* of all numbers Ω which are rational functions with integral coefficients of the given series of linearly independent real numbers $\omega_1, \omega_2, \ldots$. In his *Was sind und was sollen die Zahlen* (1888), Dedekind designates by $A \ni S$ that a system A is a part of a system S; he indicates by $\mathfrak{M}(A, B, C \ldots)$ a system made up of systems $A, B, C \ldots$ and by $\mathfrak{G}(A, B, C \ldots)$ the system composed of all elements common to $A, B, C \ldots$; he marks by $\phi(s)$ an *Abbildung* of a system S whose elements are s.

The introduction of the symbol $GF[q^n]$ to represent the Galois field of order q^n is due to E. H. Moore[4] of Chicago, in 1893.

415. *Perfect numbers.*—Perfect numbers have been called also "Euclidean numbers," and have been represented by the symbol E_p where $E_p = 2^{p-1}(2^p - 1)$, when the second factor is prime. Some confusion is likely to arise between this and the E which sometimes is made to represent "Euler's constant," or "Euler's numbers." The notation N_{prf} (*nombre parfait*) was suggested by Nassò[5] in 1900 and adopted by Peano.[6] The notation P_m for a multiply perfect number n of multiplicity m (i.e., one, the sum of whose divisors, including n and 1, is the mth multiple of n) is used by Dickson.[7]

416. *Mersenne numbers*, named after Marin Mersenne, are marked M_q by Cunningham;[8] Mersenne in 1644 asserted that numbers $M_q = 2^q - 1$, q being prime and not greater than 257, are prime only when $q = 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$.

[1] *Op. cit.*, p. 435, 436.

[2] *Op. cit.*, p. 473.

[3] G. Cantor, *Crelle's Journal*, Vol. LXXVII (1874), p. 261.

[4] E. H. Moore in *Mathematical Papers Read at the International Congress, at Chicago, 1893* (New York, 1896), p. 211. See also E. Galois, *Œuvres* (ed. E. Picard; Paris, 1897), p. 15–23.

[5] M. Nassò, *Rivista di matematica* (1900), p. 52.

[6] G. Peano, *Formulaire mathématique*, Vol. IV (1903), p. 144.

[7] L. E. Dickson, *op. cit.*, Vol. I (1919), p. 33.

[8] Allan Cunningham in *Mathematical Questions and Solutions from the Educational Times*, N.S., Vol. XIX (London, 1911), p. 81. See also L. E. Dickson, *op. cit.*, Vol. I (1919), p. 30.

417. *Fermat numbers*, $2^{2^n}+1$, are represented by F_n in Dickson's *History*.[1]

418. *Cotes's numbers* are positive fractional numbers represented by W. W. Johnson[2] by the sign ${}^nA_r$; the numbers occur in Cotes's *Harmonia Mensurarum* as coefficients of a series. Cotes gave the values of the numbers for $n=1, 2, \ldots, 10$. The values for $n=1$ are embodied in the "trapezoidal rule," and for $n=2$ in the "parabolic rule." A property of these numbers is that ${}^nA_0+{}^nA_1+\ldots+{}^nA_n=1$.

419. *Bernoulli's numbers* are given in Jakob (James) Bernoulli's *Ars conjectandi* (1713), page 97. They occur in the formulas for the sums of the even powers of the first n integers, being the coefficients of n in the formulas. Bernoulli himself computed only the first five numbers: $\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}$.

We quote the first four formulas in Bernoulli:

$$\text{“}\int n \propto \tfrac{1}{2}nn+\tfrac{1}{2}n\,.$$

$$\int nn \propto \tfrac{1}{3}n^3+\tfrac{1}{2}nn+\tfrac{1}{6}n\,.$$

$$\int n^3 \propto \tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}nn\,.$$

$$\int n^4 \propto \tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3 * -\tfrac{1}{30}n\,.\text{”}$$

Bernoulli obtains by inspection the sum for n^c. Then he states, "Literae capitales A, B, C, D etc., ordine denotant coefficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, etc. nempe $A \propto \frac{1}{6}$, $B \propto -\frac{1}{30}$, $C \propto \frac{1}{42}$, $D \propto -\frac{1}{30}$." ("The capital letters A, B, C, D, etc., denote by their order the coefficients of the last terms for $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, etc., namely, $A=\frac{1}{6}$, $B=-\frac{1}{30}$, $C=\frac{1}{42}$, $D=-\frac{1}{30}$.")

Euler[3] in 1755 used the German type of the capital letters A, B, C, D, ..., to represent the absolute values of the Bernoullian coefficients, so that $A=\frac{1}{6}$, $B=\frac{1}{30}$, etc. Euler introduced the name "Bernoullian numbers." In a paper of 1769, on the summation of series, Euler[4] considers numbers representing the products of the Ber-

[1] L. E. Dickson, *op. cit.*, Vol. I, p. 375.

[2] Roger Cotes, *Harmonia Mensurarum* (Cambridge, 1722), "De methodo differentiali Newtoniana"; W. W. Johnson's article on the Cotesian numbers is in the *Quarterly Journal of Pure and Applied Mathematics* (1914), p. 52–65.

[3] L. Euler, *Institutiones calculi differentialis* (1755), Vol. II, § 122, p. 420. See also *Euleri Opera omnia* (1st ser.), Vol. X, p. 321; Vol. XII, p. 431.

[4] *Nova Comment. Petrop. XIV* (pro 1769), p. 129–67; M. Cantor, *op. cit.*, Vol. IV (1908), p. 262.

noullian numbers $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, etc., by the numbers 6, 10, 14, 18, etc., respectively, and marks these products by the German forms of the capital letters A, B, C, D, etc. In a paper for the year 1781, Euler[1] designates the Bernoullian numbers in the same manner as he did in 1755. Euler's notation of 1755 was used by L. Mascheroni,[2] and by Gauss.[3] Von Staudt[4] employed the notation $B^{(n)}$ for the nth number; this notation occurs earlier in Klügel's *Wörterbuch*.[5]

According to G. Peano,[6] it was Euler who designated the Bernoullian numbers by the signs B_1, B_3, B_5, ... , but Peano does not give the reference. L. Saalschütz[7] says that formerly they were marked B_1, B_3, B_5, ... , as, B_2, B_4, B_6, ... , had the value zero. Scherk[8] and Grunert[9] write the first, second, ... , nth Bernoullian numbers thus: $\overset{1}{B}$, $\overset{3}{B}$, ... , $\overset{2n-1}{B}$. De Morgan[10] writes B_1, B_3, B_5, ... Binet[11] adopts the signs B_1, $-B_3$, B_5, $-B_7$, ... Others, for instance, Pascal,[12] represent these numbers by the even subscripts, B_2, B_4, B_6, ... The notation B_1, B_2, B_3, ... , which is now in wide use, was employed by Binet,[13] Ohm,[14] Raabe,[15] Stern,[16] Hermite,[17] and Adams.[18]

[1] *Acta Petrop. IV* (1781), II, p. 45; M. Cantor, *op. cit.*, Vol. IV, p. 277.

[2] L. Mascheroni, *Adnotationes ad Calculum Integralum Euleri* (Paris, 1790); *Euleri Opera omnia* (1st ser.), Vol. XII, p. 431.

[3] C. F. Gauss, *Werke*, Vol. III (Göttingen, 1866), p. 152.

[4] K. G. C. von Staudt, *Crelle's Journal*, Vol. XXI (1840), p. 373.

[5] G. S. Klügel's *Mathematisches Wörterbuch*, completed by J. A. Grunert, Vol. IV (Leipzig, 1823), "Summirung der Reihen," p. 636.

[6] G. Peano, *Formulaire mathématique*, Tome IV (Turin, 1903), § 79, p. 248.

[7] Louis Saalschütz, *Bernoullische Zahlen* (Berlin, 1893), p. 4.

[8] H. F. Scherk, *Crelle's Journal*, Vol. IV (1829), p. 299.

[9] J. A. Grunert, Supplement zu G. S. Klügel's *Wörterbuch*, Vol. I (Leipzig, 1833), "Bernoullische Zahlen."

[10] A. de Morgan, "Numbers of Bernoulli," *Penny Cyclopaedia*.

[11] J. Binet, *Comptes rendus*, Vol. XXXII (Paris, 1851), p. 920.

[12] Ernst Pascal, *Repertorium d. höheren Mathematik* (ed. A. Schepp), Vol. I (Leipzig, 1900), p. 474.

[13] J. Binet, *Journal de l'école polytechnique* (Paris,) Vol. XVI, Cahier 27 (1839), p. 240.

[14] Martin Ohm, *Crelle's Journal*, Vol. XX (1840), p. 11.

[15] J. L. Raabe, *ibid.*, Vol. XLII (1851), p. 350–51.

[16] M. A. Stern, *ibid.*, Vol. LXXXIV (1878), p. 267.

[17] C. Hermite, *ibid.*, Vol. LXXXI (1876), p. 222.

[18] J. C. Adams, *Proc. Roy. Soc. of London*, Vol. XXVII (1878), p. 88.

420. *Euler's numbers*,[1] E_{2m}, are coefficients occurring, as Ernst Pascal states, in the series $\sec x = \sum_{0}^{\infty} E_{2m} \frac{x^{2m}}{2m!}$; the name was given to these numbers by Scherk.[2] Chrystal[3] marks Euler's numbers E_1, E_2, ...; thus $E_1=1$, $E_2=5$, $E_3=61$, $E_4=1385$, etc.

SIGNS FOR INFINITY AND TRANSFINITE NUMBERS

421. The sign ∞ to signify infinite number was introduced by John Wallis in 1655 in his *De sectionibus conicis* in this manner: *Esto enim* ∞ *nota numeri infiniti*[4] (see also § 196). The conjecture[5] has been made that Wallis, who was a classical scholar, adopted this sign from the late Roman symbol ∞ for 1,000. Nieuwentijt[6] uses the letter m to represent *quantitas infinita.*

Wallis' symbol for infinity came to be used at the beginning of the eighteenth century in the Calculus. Thus in the *Acta eruditorum* for 1708 (p. 344) one encounters the oddity "$dy=\infty$, seu infinito." The same publication[7] contains "∞^n potestas indeterminate numeri infiniti" in a review of Cheyne's *Philosophical Principles of Natural Religion* (London, 1705); Cheyne[8] used this symbol freely. The symbol came to be used extensively, for instance, by Johann Bernoulli[9] in dealing with tautochrones. Sometimes, particularly in

[1] L. Euler, *Institutiones calculi differentialis* (1755), p. 542.

[2] H. F. Scherk, *Vier mathematische Abhandlungen* (Berlin, 1825). The first *Abhandlung* concerns the Eulerian and Bernoullian numbers.

[3] G. Chrystal, *Algebra*, Part II (Edinburgh, 1889), p. 318.

[4] John Wallis, *Opera mathematica*, Vol. I (Oxford, 1695), p. 297, 405; *De sectionibus conicis*, Pars I, Prop. 1; also *Arithmetica infinitorum*, Prop. 91.

[5] W. Wattenbach, *Anleitung zur latein. Paläographie* (2d ed., 1872), Appendix, p. 41.

[6] Bernhardi Nieuwentiitii, *Analysis infinitorum seu curvilineorum proprietates ex polygonorum natura deductae* (Amstelaedami, 1695). This reference is taken from H. Weissenborn, *Die Principien der höheren Analysis* (Halle, 1856), p. 124.

[7] *Acta eruditorum* (Leipzig, 1710), p. 462.

[8] See also George Cheyne, *Philosophical Principles of Religion*, Part II (London, 1716), p. 20, 21.

[9] J. Bernoulli, *Histoire de l'academie r. d. scien.*, année 1730 (Paris, 1732), *Mémoires*, p. 78; *Opera omnia*, Vol. III (1742), p. 182, 183.

writings of Euler,[1] and some later writers,[2] the symbol is not a closed figure, but simply $\backsim$. Another variation in form, due, apparently, to the exigencies of the composing-room, is seen in a book of Bangma,[3] containing "Sec $270° = 0 - 0$." B. Fontenelle, in his *Éléments de la géométrie de l'infini* (Paris, 1727), raises ∞ to fractional powers, as, for example (p. 43), "$\div 1 . \infty^{\frac{1}{4}}, \infty^{\frac{2}{4}}, \infty^{\frac{3}{4}}, \infty$." In modern geometry expressions $\infty, \infty^2, \ldots, \infty^n$ are used, where the exponents signify the number of dimensions of the space under consideration.

In the theory of functions as developed by Weierstrass[4] and his followers, the symbol ∞ is used in more than one sense. First, it is used to represent an *actual* infinity. One says that a function $f(x)$, when $\frac{1}{f(a)} = 0$, is infinite at $x = a$, or for $x = a$, and one writes $f(a) = \infty$. In this case one does not use the $+$ or $-$ signs before ∞; that is, one does not write $+\infty$ or $-\infty$. In writing $f(\infty) = b$ one means the same as in writing $\left[f\left(\frac{1}{y}\right)\right]_{y=0} = b$; in writing $f(\infty) = \infty$ one means the same as in writing $\frac{1}{\left[f\left(\frac{1}{y}\right)\right]_{y=0}} = 0$. Second, in considering the *limit of a function* $f(x)$ the $+\infty$ and $-\infty$ may arise as *virtual infinities*. For example, one has $\lim_{x=+\infty} f(x) = -\infty$, when, however large a positive constant P be taken, one can take a positive number Q, such that $f(x) < -P$ when $x > Q$.

In the theory of transfinite number, Georg Cantor[5] represents by ω an ordinal number of a rank next superior to any of the integers 1, 2, 3, ... Previously,[6] Cantor had used the sign ∞, but he discarded

[1] L. Euler in *Histoire de l'academie r. d. sciences et belles lettres*, année 1757 (Berlin, 1759), p. 311; année 1761 (Berlin, 1768), p. 203. Euler, *Institut. calculi diff.* (1755), Vol. I, p. 511, 745; Euler in *Novi Comment. acad. scient. imper. Petropolitanae*, Vol. V, for the years 1754, 1755 (Petropoli, 1760), p. 209.

[2] Matthias Hauser, *Anfangsgründe der Mathematik*, 1. Theil (Wien, 1778), p. 122.

[3] O. S. Bangma, *Verhandeling over de .... Driehoeks-meting* (Amsterdam, 1808), p. 26.

[4] K. Weierstrass, *Abh. Akad.* (Berlin, 1876), *Math.*, p. 12; *Funktionenlehre* (Berlin, 1886), p. 2; *Werke*, Vol. II, p. 78. See also A. Pringheim and J. Molk in *Encyclopédie des scien. math.*, Tom. II, Vol. I, p. 20, 31, 32 (1909).

[5] Georg Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre* (Leipzig, 1883), p. 33.

[6] Georg Cantor, *Mathematische Annalen*, Vol. XVII (1880), p. 357.

that in 1882, in favor of ω, "because the sign ∞ is used frequently for the designation of undetermined infinities."[1] Cantor's first ordinal of the second class of numbers (II) is therefore ω; he[2] marks the first ordinal of the third class (III) by Ω; Bertrand Russell[3] marks the first of the $(\nu+2)$th class by ω_ν, so that ω_1 is the same as Ω.

Cantor designates by the sign aleph-zero $\aleph_0$, which is the Hebrew letter aleph with a subscript zero attached, the smallest transfinite *cardinal* number,[4] and by $\aleph_1$ the next superior cardinal number. Peano,[5] in 1895, indicated cardinal numbers by the abbreviation Nc.

Earlier than this the aleph had been used as a symbol for 60 in the Hebrew numeral notation, and it was used as a mathematical symbol for certain fundamental functions by H. Wronski,[6] in his *Philosophie des mathématiques* (1811).

The cardinal number of the aggregate constituting all numbers in the linear continuum[7] is marked $\mathfrak{c}$. The cardinal number of the aggregate F of all functions[8] of a real variable (known to be greater than $\mathfrak{c}$) is marked $\mathfrak{f}$.

The different alephs of well-ordered transfinite aggregates are marked $\aleph_0\ \aleph_1 \ldots\ldots \aleph_\nu, \ldots\ldots, \aleph_\omega, \ldots\ldots, \aleph_\alpha$, and Schoenflies[9] writes the initial numbers of each class of ordinals, $\Omega_0, \Omega_1, \ldots\ldots, \Omega_\nu, \ldots\ldots, \Omega_\omega, \ldots\ldots, \Omega_d$, so that $\Omega_0=\omega$ and $\Omega_1=\Omega$, which latter is G. Cantor's designation for the first number in his third class, and is sometimes[10] written also ω_1. G. Cantor[11] marked a derived aggregate of μ by μ'; Peano marked it Du in 1895[12] and δu in 1903.

Other symbols used by G. Cantor[13] in his theory of aggregates are

[1] *Op. cit.*, Vol. XXI (1883), p. 577.

[2] *Op. cit.*, Vol. XXI, p. 582.

[3] B. Russell, *Principles of Mathematics*, Vol. I (1903), p. 322.

[4] G. Cantor, *Mathematische Annalen*, Vol. XLVI (1895), p. 492.

[5] G. Peano, *Formulaire de mathématiques*, Vol. I (1895), p. 66; Vol. IV (1903), p. 128.

[6] Gergonne's *Annales de mathématiques* (Nismes), Vol. III (1812, 1813), p. 53.

[7] A. Schoenflies, *Entwickelung der Mengenlehre und ihrer Anwendungen* (1913), p. 54; E. V. Huntington, *The Continuum* (2d ed., 1917), p. 80.

[8] A. Schoenflies, *op. cit.*, p. 60.

[9] A. Schoenflies, *op. cit.*, p. 124.

[10] E. V. Huntington, *op. cit.* (2d ed., 1917), p. 72.

[11] G. Cantor, *Mathematische Annalen*, Vol. V, p. 123.

[12] G. Peano, *Formulaire mathématique*, Vol. I (1895), p. 69; Vol. IV (1903), p. 121.

[13] G. Cantor, *Mathematische Annalen*, Vol. XLVI (1895), p. 481–512; see also P. E. B. Jourdain's translation of this article, and the one in Vol. XLIX (1897), p. 207–46, in a book, *Contributions to the Founding of the Theory of Transfinite Numbers* (Chicago and London, 1915).

$M=\{m\}$, where M suggests *Menge* ("aggregate"); $(M, N, P, \ldots.)$ the uniting of the aggregates $M, N, P, \ldots.$, which have no common elements into a single aggregate; $\overline{\overline{M}}$ the general symbol for *Mächtigkeit* ("power") or "cardinal number" of M; when two aggregates M and N are "equivalent," Cantor writes $M \sim N$. For two aggregates M and N, the cardinal numbers are designated also by the signs $\mathfrak{a}=\overline{\overline{M}}$, $\mathfrak{b}=\overline{\overline{N}}$. The "covering-aggregate (*Belegungsmenge*) of N with M" is denoted by $(N|M)$, and $\mathfrak{a}^{\mathfrak{b}}=\overline{\overline{(N|M)}}$.

If in a simply ordered aggregate $M=\{m\}$, the element m_1 has a rank lower than m_2 in the given order of precedence, this relation is expressed $m_1 \prec m_2$, $m_2 \succ m_1$. If the infinite aggregates are multiply ordered F. Riesz[1] used the symbolism $a \underset{i}{=} b$, $a <i\, b$, $a\, i> b$ to designate that in the ith order, a and b have equal rank, or a precedes b, or a follows b. If two ordered aggregates M and N are "similar," they are marked by Cantor $M \simeq N$; the "ordinal type" of M is marked $\overline{M}$. If α is the ordinal type, the corresponding cardinal number is $\bar{\alpha}$. If in an ordered aggregate M all the relations of precedence of its elements are inverted, the resulting ordered aggregate is denoted by $^*M$ and its ordinal type $^*\alpha$, provided $\alpha=\overline{M}$. Cantor[2] designates a linear continuous series having both a first and a last element, a series of type θ.

When two fundamental series $\{a_\nu\}$ and $\{a'_\nu\}$ of numbers of the first or second number-class are *zusammengehörig* ("coherent")[3] they are marked $\{a_\nu\} \| \{a'_\nu\}$. The "ϵ-numbers of the second number-class" are roots of the equation $\omega^\xi=\xi$. The symbol $E(\gamma)$ represents the limit of the fundamental series $\{\gamma_\nu\}$, and is an ϵ number.

G. Cantor[4] in 1880 marked the least common multiple of aggregates M and N by $\mathfrak{M}\{M, N\}$, but E. Zermelo[5] and A. Schoenflies[6] adopted the sign $\mathfrak{S}(M, N)$ suggested by the first letter in *Summe* ("sum"). The *Durchschnitt*, or all the elements common to M and N, are marked $[M, N]$ by Zermelo and $\mathfrak{D}(M, N)$ by Cantor and Schoenflies.[7] The sign C_ν is used to represent the continuum[8] of space of ν

[1] F. Riesz, *Mathematische Annalen*, Vol. LXI (1903), p. 407.

[2] G. Cantor, *Mathematische Annalen*, Vol. XLVI (1895), p. 511.

[3] G. Cantor, *ibid.*, Vol. XLIX (1897), p. 222.

[4] G. Cantor, *ibid.*, Vol. XVII (1880), p. 355.

[5] E. Zermelo, *Mathematische Annalen*, Vol. LXV (1908), p. 265.

[6] A. Schoenflies, *Entwickelung der Mengenlehre und ihrer Anwendungen* (Leipzig und Berlin, 1913), p. 10.

[7] A. Schoenflies, *op. cit.*, p. 11. [8] A. Schoenflies, *op. cit.*, p. 11, 56.

dimensions. If $X_a = \Sigma x_a$ are ordered aggregates, their product yields an ordered product-aggregate which F. Hausdorff[1] represents by $\overset{A}{\underset{a}{\Pi}} Xa$ and Schoenflies[2] by $^*\Pi Xa$; if such a product is a *Vollprodukt* ("complete product"), Hausdorff (writing Ma for Xa) marks it $((\overset{A}{\underset{a}{\Pi}} Ma))$ and its type by $((\overset{a}{\underset{a}{\Pi}} \mu a))$. He designates the "maximal product" by T.

In the theory of the equivalence of aggregates E. Zermelo[3] writes $\phi \in MN$, to express the *Abbildung* of M upon N, where M and N are aggregates having no elements in common and ϕ is a subaggregate such that each element of $M+N$ appears as element in one and only one element $\{m, n\}$ of ϕ.

Peano[4] designates by $a \mathbin{!} b$ the aggregate of all the couples formed by an object of a class a with an object of class b, while the sign $a;b$ designates the couple whose two elements are the classes a and b. J. Rey Pastor[5] designates the existence of a one-to-one correspondence between infinite aggregates A and B, by the symbol $A \barwedge B$.

SIGNS FOR CONTINUED FRACTIONS AND INFINITE SERIES

422. *Continued fractions.*—We previously mentioned (§ 118) a notation for continued fractions, due to al Ḥaṣṣâr, about the beginning of the thirteenth century, who designates[6] by $\frac{3}{5}\frac{3}{8}\frac{2}{9}$ the ascending fraction

$$\frac{2+\frac{3+\frac{3}{5}}{8}}{9}.$$

A Latin translation of the algebra of the famous Arabic algebraist and poet, Abu Kamil,[7] contains a symbolism for ascending continued

[1] F. Hausdorff, *Mathematische Annalen*, Vol. LXV (1908), p. 462.

[2] A. Schoenflies, *op. cit.*, p. 75–77.

[3] E. Zermelo, *Mathematische Annalen*, Vol. LXV (1908), p. 267, 268.

[4] G. Peano, *Formulaire mathématique*, Tome IV (1903), p. 125.

[5] *Revista matematica Hispano-Americana*, Vol. I (Madrid, 1919), p. 28.

[6] H. Suter in *Bibliotheca Mathematica* (3d ser.), Vol. II (1901), p. 28.

[7] Paris MS 7377 A, described by L. C. Karpinski, *Bibliotheca mathematica* (3d ser.), Vol. XII (1911–12). Our reference is to p. 53. See also Vol. X (1909–10), p. 18, 19, for H. Suter's translation into German of a translation into Italian of a Hebrew edition of Abu Kamil's book, *On the Pentagon and Decagon*, where $\frac{6}{8}\text{-}\frac{5}{8}$ signifies $\frac{6}{8}$ plus $\frac{5}{8}$ of $\frac{1}{8}$, or $\frac{53}{64}$.

fractions, which differs from that of al-Ḥaṣṣâr in proceeding from left to right. The fraction $\frac{1\ 1}{9\ 9}$ stands for $\frac{1}{9}$ plus $\frac{1}{81}$; the fraction $\frac{0\ 1}{9\ 2}$ or $\frac{0\ 1}{9\ 2}$ stands for $\frac{1}{2}$ of $\frac{1}{9}$. Abu Kamil himself presumably proceeded from right to left. The Arabic notation is employed also by Leonardo[1] of Pisa in his *Liber abbaci* of 1202 and 1228, and later still by the Arabic author al-Qualasâdî.[2] Al-Ḥaṣṣâr, it will be observed, also used the fractional line for ordinary fractions, as did also Leonardo of Pisa.

423. Leonardo[3] gives expressions like $\frac{7\ \ 1}{10\ 10}\,8$, which must be read from right to left and which stands in our notation for $8+\frac{1+\frac{7}{10}}{10}$ or $8\frac{17}{100}$. He gives also $\frac{1\ \ 0\ \ \ 0}{5\ 12\ 20}$ as equal to $\frac{1\ 0\ \ 0\ \ \ 0}{2\ 6\ 10\ 10}$; the value of these ascending continued fractions is $\frac{1}{1200}$.

Leonardo gives also two other notations, one of which he describes as follows: "Et si in uirga fuerint plures rupti, et ipsa uirga terminauerit in circulo, tunc fractiones eius, aliter quam dictum sit denotabunt, ut in hac $\cdot\frac{2\,4\,6\,8}{3\,5\,7\,9}\bigcirc$ cuius uirge fractiones denotant, octo nonas unius integri, et sex septimas de octononis, et quatuor quintas sex septimarum de octo nonis et duas tertias quattuor quintarum sex septimarum de octo nonis unius integri.[4] ("And if on the line there should be many fractions and the line itself terminated in a circle, then its fractions would denote other than what has been stated, as in this $\cdot\frac{2\,4\,6\,8}{3\,5\,7\,9}\bigcirc$, the line of which denotes the fractions eight-ninths of a unit, and six-sevenths of eight-ninths, and four-fifths of six-sevenths of eight-ninths, and two-thirds of four-fifths of six-sevenths of eight-ninths of an integral unit.") We have here an ascending continued fraction

$$\frac{8+\frac{48+\frac{192+\frac{384}{3}}{5}}{7}}{9}\,.$$

Leonardo's third notation is described by him as follows: "Item si uirgule protraherunter super uirgam in hunc modum $\frac{1\,1\,1\,5}{5\,4\,3\,9}$, denotant fractiones eius quinque nonas et tertiam et quartam et quintam unius none."[5] ("Also if a short line be drawn above the

[1] Leonardo of Pisa, *Scritti* (pub. by B. Boncompagni), Vol. I (Rome, 1857), p. 24. See also *Encyclopédie des scienc. math.*, Tome I, Vol. I (1904), p. 318, n. 330.

[2] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I (3d ed., 1907), p. 813.

[3] Leonardo of Pisa, *op. cit.*, Vol. I, p. 85.

[4] *Op. cit.*, Vol. I (1857), p. 24.

[5] *Op. cit.*, Vol. I (1857), p. 24, also p. 91, 92, 97. See G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XII (1911–12), p. 62.

fractional line in this manner $\frac{1\,1\,1\,5}{5\,4\,3\,9}$, they denote the following fractions, five-ninths, and one-third, one-fourth and one-fifth of one-ninth.") The foregoing fraction signifies, therefore, $\frac{5+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{9}$; it is a complex fraction, but hardly of the "continued" type. Not to be confounded with these is Leonardo's notation $\circ\frac{8\,6\,4\,2}{9\,7\,5\,3}$ which does not represent a continued fraction either, but signifies the multiplication[1] of the fractions, thus $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}$, and resembles a mode of writing, $\frac{3}{7}$—$\frac{3}{5}$, for $\frac{3}{5}\cdot\frac{3}{7}$, sometimes found in al-Ḥaṣṣâr.[2]

424. Pietro Antonio Cataldi in 1606[3] discusses the ascending continued fraction as "vna quãtità scritta, ò proposta in forma di rotto di rotto" ("a quantity written or proposed in the form of a fraction of a fraction"). He explains an example and tells how mathematicians are accustomed to write it: "Sogliono i Pratici scriuere cosi $\frac{3}{8}\cdot\frac{2}{5}\cdot\frac{6}{9}\cdot\frac{4}{5}\cdot\frac{5}{8}$." The meaning of this appears from a simpler example explained on the previous page (142). He says: "poniamo 3. quarti, & $\frac{1}{2}$. quarto, cioè, $\frac{3}{4}$.& $\frac{1}{2}$." or $\frac{7}{8}$. One has here a notation for an ascending continued fraction. Seven years later Cataldi introduced, for the first time, a notation for descending continued fractions, and he chose practically the symbolism that he had used for the ascending ones. In 1613 he explained: "Notisi, che nõ si potendo cõmodamẽte nella stampa formare i rotti, & rotti di rotti come andariano cioè così

$$4.\&\frac{2}{8.\&\frac{2}{8.\&\frac{2}{8}}}$$

come ci siamo sforzati di fare in questo, noi da qui inãzi gli formaremo tutti à q̃sta similitudine

$$4.\&\frac{2}{8.}\ \&\frac{2}{8.}\ \&\frac{2}{8.}$$

facendo vn punto all '8. denominatore de ciascum rotto, à significare, che il sequente rotto è rotto d'esso denominatore."[4] ("Observe that

[1] Leonardo of Pisa, *op. cit.*, Vol. I, p. 24.

[2] H. Suter, *Bibliotheca mathematica* (3d ser.), Vol. II, p. 27.

[3] Pietro Antonio Cataldi, *Seconda parte della pratica aritmetica* (Bologna, 1606), p. 143.

[4] *Trattato del Modo Brevissimo di trouare la Radice quadra delli numeri, et Regole da approssimarsi di continuo al vero nelle Radici de' numeri non quadrati* ... Di Pietro Antonio Cataldi Lettore delle Scienze Mathematiche nello Studio di Bologna (in Bologna, 1613), p. 70, 71.

since in printing when proceeding hurriedly, one cannot conveniently form fractions, and fractions of fractions, in this form

$$4.\ \&\frac{2}{8.\ \&\frac{2}{8.\ \&\frac{2}{8}}}$$

as here we are forcing ourselves to do it, we can easily denote all of them by adopting this imitation

$$4.\ \&\frac{2}{8.}\ \&\frac{2}{8.}\ \&\frac{2}{8.}$$

placing a point after the denominator 8. in each fraction, to signify that the fraction following is a fraction of the denominator.") But, in his simplified form of writing, as it appears further on in his book, the point is not placed after the numbers in the denominator, but is placed higher up, on a level with the "&." This is seen in Cataldi's approximation to $\sqrt{18}$ (p. 71):

"Di 18. la ℞ sia $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}$
ò vogliamo dire $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{1}{4}$
cioè $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{8}{33}$
ò vogliamo dire $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{1}{4}\ \tfrac{4}{33}$
che è $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{33}{136}$
cioè $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{2}{8}\cdot\&\ \tfrac{272}{1121}$
ò vogliamo dire $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{1}{4}\ \tfrac{136}{1121}$
che è $4.\&\ \tfrac{2}{8}\cdot\&\ \tfrac{1121}{4620}$
cioè $4.\&\ \tfrac{9240}{38081}$."

Except for the "&" in place of "+," this notation conforms closely with one of the modern notations.

425. We come next to Wallis' representation of Brounker's expression[1] for $\frac{4}{\pi}$, namely,

$$\text{“}\ \square = 1\tfrac{1}{2}\ \tfrac{9}{2}\ \tfrac{25}{2}\ \tfrac{49}{2}\ \tfrac{81}{2}\ \&\ \text{c.,” and Wallis'}\qquad \text{“}\ \frac{a}{\alpha}\ \frac{b}{\beta}\ \frac{c}{\gamma}\ \frac{d}{\delta}\ \frac{e}{\epsilon}\ \&\text{c.”}$$

[1] John Wallis, *Arithmetica infinitorum* (Oxford, 1655), p. 182.

The second expression is Wallis' general representation[1] of a continued fraction. The omission of the signs of addition makes this notation inferior to that of Cataldi who used the "&." But Wallis adheres to his notation in his collected works[2] of 1695, as well as in his letter[3] to Leibniz of April 6, 1697. Leibniz,[4] on the other hand, used an improved form,

$$\text{“}\, a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.”}}}}}$$

in his letter to John Bernoulli of December 28, 1696.

A modern symbolism is found also in C. Huygens[5] who expressed the ratio 2640858:77708431 in the form

$$\text{“}29+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{4}\ \text{etc.”}}}}}}$$

The symbolism of Leibniz and Huygens came to be the regular form adopted by eighteenth-century writers and many writers of more recent date; as, for example, by J. A. Serret[6] in his *Higher Algebra.* Among eighteenth-century writers we cite especially L. Euler and J. Lagrange.

426. L. Euler[7] writes

$$\text{“}\, a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}+\text{etc.”}}}$$

In some of his articles the continued fractions written in this notation made extraordinary demands upon space, indeed as much as

[1] *Op. cit.*, p. 191.

[2] John Wallis, *Opera mathematica*, Vol. I (1695), p. 469, 474.

[3] C. I. Gerhardt, *Leibnizens Mathematische Schriften*, Vol. IV (Halle, 1859), p. 17.

[4] *Op. cit.*, Vol. III (Halle, 1855), p. 351, 352.

[5] Christian Huygens, *Descriptio automati planetarii* (The Hague, 1698); *Opuscula posthuma* (Amsterdam, 1728), Vol. II, p. 174–79. See also S. Günther, in *Bullettino Boncompagni*, Vol. VII (Rome, 1874), p. 247, 248.

[6] J. A. Serret, *Cours d'algèbre supérieure* (Paris, 1854), p. 491; German edition by G. Wertheim, Vol. I (2d ed.; Leipzig, 1878), p. 8.

[7] L. Euler, *Introductio in analysin infinitorum* (1748); ed. 1922, Vol. I, p. 362.

13 cm. or four-fifths of a large page.[1] In 1762 he devised a new notation in his *Specimen Algorithmi Singularis;*[2] a fraction

$$a+\frac{1}{b}+\frac{1}{c}$$

is represented by the symbol $\frac{(a, b, c)}{(b, c)}$; an infinite continued fraction by $\frac{(a, b, c, d, e \text{ etc.})}{(b, c, d, e \text{ etc.})}$. This symbolism offers superior advantages in computation, as, for example: $(a, b, c, d, e)=e(a, b, c, d)+(a, b, c)$, also $(a, b, c, d, e)=(e, d, c, b, a)$.

427. E. Stone[3] writes

$$\text{“}\frac{314159}{100000} \text{ will be}=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{7+\frac{1}{4}}}}}}}\text{.”}$$

This extension of the fractional lines to the same limit on the right is found earlier in Leibniz (§ 562) and is sometimes encountered later, as, for instance, in the book on continued fractions by O. Perron (1913).[4]

428. In the nineteenth century the need of more compact notations asserted itself and there was a return to Cataldi's practice of placing all the partial fractions on the same level. Sir John W. Herschel[5] writes the continued fraction

$$\frac{c_1}{a_1}+\frac{c_2}{a_2}+\frac{c_3}{a_3}+\ldots\ldots\frac{c_x}{a_x}$$

in the form

$$\frac{c_1}{a_1+}\,\frac{c_2}{a_2+}\,\frac{c_3}{a_3+}\,\ldots\ldots\frac{c_x}{a_x},$$

[1] See L. Euler in *Novi comment. academ. imper. Petropolitanae,* Vol. V, for the years 1754, 1755 (Petropoli, 1760), p. 225, 226, 231.

[2] *N. Comm. Petr. IX, pro annis 1762 et 1763* (Petropoli, 1764), p. 53–69. See M. Cantor, *op. cit.*, Vol. IV (1908), p. 155.

[3] E. Stone, *New Mathematical Dictionary* (2d ed.; London, 1743), art. "Ratio."

[4] Oskar Perron, *Lehre von den Kettenbrüchen* (Leipzig und Berlin, 1913), p. 3.

[5] J. F. W. Herschel, *Collection of Examples of . . . . Calculus of Finite Differences* (Cambridge, 1820), p. 148.

but states that this is "after the example" of Heinrich Bürmann of Mannheim, a worker in C. F. Hindenburg's school of combinatory analysis. The notation of Bürmann and Herschel has been used in England by Hall and Knight,[1] C. Smith,[2] G. Chrystal,[3] L. Ince,[4] and is still widely used there. Chrystal (Part II, p. 402) also uses symbolisms, such as:

$$\sqrt{13}=3+\underset{*}{\frac{1}{1+}}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\underset{*}{\frac{1}{6+}}\,\frac{1}{1+}\cdots,$$

where the * * indicate the beginning and end of the cycle of partial quotients.

429. Möbius[5] says: "Der Raum-Ersparniss willen mögen nun die Kettenbrüche von der besagten Form, wie

$$\frac{1}{a},\quad \frac{1}{a-\frac{1}{b}},\quad \frac{1}{a-\frac{1}{b-\frac{1}{c}}},\quad \frac{1}{a-\frac{1}{b-\frac{1}{c-\frac{1}{d}}}}$$

durch (a), (a, b), (a, b, c), (a, b, c, d), u.s.w. ausgedrückt werden." Möbius[6] represents the function

$$\frac{1}{(a,\ldots . e)(b,\ldots . e)(c, d, e)(d, e)(e)}$$

by the symbolism $[a, b, c, d, e]$. M. Stern[7] designated the continued fraction

$$a+\frac{b_1}{a_1}+\frac{b_2}{a_2}+\vdots+\frac{b_m}{a_m}$$

[1] H. S. Hall and S. R. Knight, *Elementary Algebra* (ed. F. L. Sevenoak; New York, 1895), p. 369.

[2] C. Smith, *Elementary Algebra* (ed. Irving Stringham; New York, 1895), p. 530.

[3] G. Chrystal, *Algebra*, Part II (Edinburgh, 1889), p. 396, 397.

[4] Linsay Ince in *University of Edinburgh Mathematical Dept.* (1914), Paper No. 7, p. 2, 3.

[5] A. F. Möbius, *Crelle's Journal*, Vol. VI (1830), p. 217; *Werke*, Vol. IV (Leipzig, 1887), p. 507.

[6] Möbius, *Crelle's Journal*, Vol. VI, p. 220.

[7] M. Stern, *Theorie der Kettenbrüche und ihre Anwendung* (Berlin, 1834), p. 4, 22, 33; reprinted from *Crelle's Journal*, Vol. X, XI.

by $F(a, a_m)$; he lets a_1, a_m stand symbolically for the denominator of the equivalent ordinary fraction, and a, a_m for its numerator, so that $F(a, a_m) = \dfrac{a, a_m}{a_1, a_m}$. Stern employs also the fuller form $F(a, a_m) = F(a + b_1 : a_1 + b_2 : a_2$, etc.). For the special, infinite continued fraction $F(1:1+1:2+9:2+25;2$, etc.) he suggests the form

$$\underset{0-\infty}{x} F[1:1+(2x+1)^2:2] \, .$$

430. J. H. T. Müller[1] devised the notation

$$b_0 + \frac{a_1}{b_1} \dot{+} \frac{a_2}{b_2} \dot{+} \cdots \dot{+} \frac{a_n}{b_n} ,$$

where each + sign may be replaced by a − sign when parts are negative. Müller also wrote

$$b_0 + \left(\frac{a}{b}\right)_{(1 \dot{+} 2 \dot{+} 3 \dot{+} \ldots \dot{+} n)} .$$

Müller's first symbolism found quite wide acceptance on the European continent, as is born out by statements of Chrystal[2] and Perron.[3] Perron remarks that sometimes the dots were omitted, and the continued fraction written in the form

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} .$$

431. When all the numerators of the partial fractions are unity, G. Lejeune Dirichlet[4] wrote down simply the denominators b_r, in the following form:

$$(b_0, b_1, \ldots, b_{n-1}, b_n).$$

E. Heine[5] writes the continued fraction

$$\lambda_0 - \frac{\mu_1}{\lambda_1 - \dfrac{\mu_2}{\lambda_2 - \cdots \dfrac{\mu_n}{\lambda_n}}}$$

[1] J. H. T. Müller, *Lehrbuch der Mathematik. Erster Theil, Die gesammte Arithmetik enthaltend* (Halle, 1838), p. 384.

[2] G. Chrystal, *Algebra*, Part II (Edinburgh, 1889), p. 397.

[3] O. Perron, *op. cit.*, p. 3.

[4] G. Lejeune Dirichlet, *Abhandl. Akad. Berlin* (1854), *Math.*, p. 99; *Werke*, Vol. II (Berlin, 1897), p. 141.

[5] E. Heine, *Theorie der Kugelfunctionen* (Berlin, 1878), p. 261.

in the form

$$\left|\begin{array}{cccccc} & \mu_1 & \mu_2 & \ldots\ldots & \mu_{n-1} & \mu_n \\ \lambda_0 & \lambda_1 & \lambda_2 & \ldots\ldots & \lambda_{n-1} & \lambda_n \end{array}\right| .$$

This notation is followed by Pascal[1] in his *Repertorium.*

432. The modern notation, widely used on the European continent,

$$b_0 \pm \frac{a_1|}{|b_1} \pm \frac{a_2|}{|b_2} \pm \ldots\ldots \pm \frac{a_n|}{|b_n} ,$$

is due to Alfred Pringsheim,[2] who represents this continued faction also by the symbol

$$\left[b_0; \pm \frac{a_\nu}{b_\nu}\right]_1^n ;$$

by

$$\left[b_m; + \frac{a_\nu}{b_\nu}\right]_{m+1}^n$$

he represents the continued fraction whose first term is b_m and the denominator of the first partial fraction, b_{m+1}. Also in place of

$$\left[0; - \frac{a_\nu}{b_\nu}\right]_{m+1}^n$$

he writes simply

$$\left[- \frac{a_\nu}{b_\nu}\right]_{m+1}^n .$$

G. Peano[3] represents the continued fraction

$$\frac{1}{a_1 + \frac{1}{a_2 + \text{etc. to } a_n}} ,$$

by the symbolism $F_c(a_1, 1 \ldots\ldots n)$.

In the continued fraction

$$a_1 + \frac{b_2}{a_2+} \frac{b_3}{a_3+} \ldots\ldots \frac{b_n}{a_n} ,$$

[1] E. Pascal, *Repertorium der höheren Mathematik* (ed. P. Epstein and H. E. Timerding), Vol. I (2d ed., 1910), p. 444.

[2] Alfred Pringsheim, *Encyklopädie der mathematischen Wissenschaften*, 1. Band, 1. Theil (Leipzig, 1898–1904), p. 119. These notations occur also in the French edition, the *Encyclopédie des scien. math.*, Tome I, Vol. I (1904), p. 283, 284.

[3] G. Peano, *Formulaire mathématique*, Vol. IV (1903), p. 197.

the expression $p_n = a_n p_{n-1} + b_n p_{n-2}$, where $p_0 = 1$, $p_1 = a_1$, is called by Chrystal[1] a "continuant of the nth order" and is marked

$$p_n = K\begin{pmatrix} & b_2, \ldots\ldots, b_n \\ a_1, & a_2, \ldots\ldots, a_n \end{pmatrix},$$

a symbol attributed[2] to Thomas Muir.

433. A notation for ascending continued fractions corresponding to Bürmann and Herschel's notation for descending ones is the following,

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \ldots\ldots + \frac{b_n}{a_n},$$

used, among others, by Weld.[3] A. Pringsheim and J. Molk[4] write an ascending continued fraction

$$K^{(n)} = \cfrac{a_1 + \cfrac{a_2 + \cdots + \cfrac{a_n}{b_n}}{b_2}}{b_1}$$

in the form

$$K^{(n)} = \left\{\frac{a_\nu}{b_\nu}\right\}_1^n .$$

434. *Tiered fractions.*—G. de Longchamps[5] indicates by

$$A_n = \left| \begin{array}{c} a_1 \\ \hline a_2 \\ \hline \vdots \\ a_n \\ \hline a_{n+1} \end{array} \right|$$

tiered fractions (*fractions étagées*) involving the numbers $a_1, a_2, \ldots\ldots, a_{n+1}$, the bars between the numbers signifying division. If the bars be assigned different lengths, A_n has a definite arithmetical value. Thus,

$$\cfrac{\cfrac{a_1}{a_2}}{\cfrac{a_3}{a_4}} = \frac{a_1 a_3 a_4}{a_2}\ ; \qquad \cfrac{\cfrac{a_1}{a_2}}{\cfrac{a_3}{a_4}} = \frac{a_1}{a_2 a_3 a_4} .$$

[1] G. Chrystal, *Algebra*, Part II (1889), p. 466. [2] O. Perron, *op. cit.*, p. 6.

[3] Laenas G. Weld, *Theory of Determinants* (2d ed.; New York, 1896), p. 186.

[4] A. Pringsheim and J. Molk, *Encyclopédie des scien. math.*, Tome I, Vol. I (1904), p. 317.

[5] Gobierre de Longchamps, *Giornale di matem.* (1st ser.), Vol. XV (1877), p. 299.

A_n may have $n!$ different values, arising from the different relative lengths which may be assigned to the fractional lines.

435. *Infinite series.*[1]—The early writers in the field of infinite series or infinite products indicated such expressions by writing down the first few terms or factors and appending "&c." Thus Wallis in 1656 wrote: "1, 6, 12, 18, 24, &c.,"[2] and again "$1\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}$, &c.,"[3] and similarly for infinite continued fractions.[4] He uses the "&c." also for finite expressions, as in "$0+a^3+b^3+c^3$ &c. cujus terminus ultimus l^3, numerus terminorum $l+1$"[5] (whose last term [is] l^3, the number of terms $l+1$).

Nicholas Mercator writes "$ps=1-a+aa-a^3+a^4$, &c.,"[6] and also "$\frac{a}{b}+\frac{ca}{bb}+\frac{cca}{b^3}+\frac{c^3a}{b^4}$, & deinceps continuando progressionem in infinitum."[7] James Gregory, in the same year, says: "Si fuerint quantitates continuè proportionales, A, B, C, D, E, F, &c. numero infinitae."[8] He places the sign $+$ before "&c." when the algebraic sum of the terms is expressed, as in "erit primus terminus $+\frac{1}{2}$ secundi $+\frac{1}{3}$ tertii $+\frac{1}{4}$ quarti $+\frac{1}{5}$ quinti $+$&c. in infinitum $= =$ spatio Hyperbolico S B KH."[9] In other passages he leaves off the "in infinitum."[10] Brounker[11] in 1668 writes

$$\text{“}\frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\frac{1}{7\times8}+\frac{1}{9\times10}, \text{ \&c. in infinitum.”}$$

G. W. Leibniz[12] gives the quadrature of the circle by the series $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}$, and ends with "$+\frac{1}{17}$ &c." Cotes[13] writes

$$\text{“}v*-\tfrac{1}{6}v^3+\tfrac{5}{24}v^4-\tfrac{23}{60}v^5, \text{ \&c.”}$$

Wolff[14] says "$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$ und so weiter unendlich fort."

[1] See also § 408.

[2] John Wallis, *Arithmetica infinitorum* (Oxford, 1655), p. 26.

[3] *Op. cit.*, p. 175. [4] *Op. cit.*, p. 191. [5] *Op. cit.*, p. 145.

[6] *Logarithmo-technia . . . . auctore Nicolao Mercatore* (London, 1668), Propositio XVII, p. 32.

[7] *Op. cit.*, p. 25.

[8] James Gregory, *Exercitationes geometricae;* in the part on Mercator's quadrature of the hyperbola, p. 9.

[9] James Gregory, *op. cit.*, p. 11. [10] James Gregory, *op. cit.*, p. 13.

[11] L. Viscount Brounker, *Philosophical Transactions* (London), Vol. II, p. 645–49; abridged edition by John Lowthrop, Vol. I (London, 1705), p. 10.

[12] *Philosoph. Transactions,* abridged, Vol. I (London, 1705), p. 16. The practice of Leibniz is exhibited also in articles in the *Acta eruditorum* (1691), p. 179; (1693), p. 178; (1694), p. 369; (1695), p. 312; (1701), p. 215.

[13] Roger Cotes, *Harmonia mensurarum* (Cambrigiae, 1722), p. 6.

[14] Christian Wolff, *Mathematisches Lexicon* (Leipzig, 1716), p. 176.

436. T. Watkins[1] uses dots in place of "&c." to represent the tail end of a series, but he does not write the + or the − after the last term written down. In a letter to Nikolaus I. Bernoulli, May 14, 1743, Euler[2] writes $1+x+x^2+x^3+\ldots\ldots+x^\infty$, and also $1-3+5-7+9-\ldots\ldots\pm(2\infty+1)$.

E. Waring[3] writes series thus, "$\alpha+\beta+\gamma+\delta+\epsilon+\zeta+$etc."

The indefinite continuance of terms is designated by Schultz[4] in this manner, "$1+3+5+7+\ldots\ldots\sim$," the $\sim$ being probably intended for Wallis' ∞. Owing no doubt to the lack of proper type in printing offices Wallis' symbol was given often forms which stood also for "difference" or "similar." Thus the sign ∽ stands for "infinity" in a publication of F. A. Prym[5] in 1863.

The use of dots was resorted to in 1793 by Prändel[6] when he writes "□ Circ.$=4-\frac{2}{3}-\frac{1}{10}-\frac{1}{28}-\frac{1}{288}\ldots\ldots$" L'Huilier[7] ends an infinite series of positive terms with "$+\ldots\ldots$"

L'Abbé de Gua[8] writes a finite expression, marking the terms omitted, with dots and also with "&c.," "3, 4, 5 .... &c $\overline{n-m+2}$," the commas indicating here multiplication. F. Nicole[9] writes a procession of factors, using dots, but omitting the "&c." and also the bar: "$n\times n-1\times\ldots\ldots n-7$." The same course is pursued by Condorcet.[10] On the other hand, C. A. Vandermonde[11] used the "&c." as in "$a+b+c+$&c." Paulus Mako[12] of Austria designates that the series is infinite, by writing "b, bm, bm^2, bm^3, ... bm^∞." C. F. Hindenburg[13] uses dots between, say, the fourth term and the nth

[1] Thomas Watkins, *Philosophical Transactions*, Vol. XXIX (1714–16), p. 144.

[2] L. Euler, *Opera posthuma*, I (Petropoli, 1862), p. 538. See G. Eneström in *Bibliotheca mathematica* (3d ser.), Vol. XI (1910–11), p. 272.

[3] E. Waring, *Meditationes algebraicae* (Cantibrigiae; 3d ed., 1782), p. 75.

[4] Johann Schultz, *Versuch einer genauen Theorie des Unendlichen* (Königsberg, 1788), p. xxvii.

[5] F. A. Prym, *Theoria nova functionum ultraellipticarum* (Berlin, 1863), p. 1, 3, 4.

[6] Johann Georg Prändel, *Kugldreyeckslehre u. höhere Mathematik* (München, 1793), p. 37.

[7] Simon l'Huilier, *Principiorum Calculi Differentialis et Integralis Expositio* (Tubingae, 1795), p. 27.

[8] De Gua in *Histoire de l'académie r. d. sciences*, année 1741 (Paris, 1744), p. 81.

[9] F. Nicole, *op. cit.*, année 1743 (Paris, 1746), p. 226.

[10] Le Marquis de Condorcet, *op. cit.*, année 1769 (Paris, 1772), p. 211.

[11] C. A. Vandermonde, *op. cit.*, année 1771 (Paris, 1774), p. 370.

[12] *Compendiaria matheseos institutio .... Paulus Mako* (3d ed.; Vienna, 1771), p. 210.

[13] C. F. Hindenburg, *Infinitinomii dignitatum .... historia leges ac formulae .... auctore Carolo Friderico Hindenburg* (Göttingen, 1779), p. 5, 6, 41.

term, the + or the − sign being prefixed to the last or nth term of the polynomial. However, at times he uses "&c." in place of the dots to designate the end of a polynomial, "$cC+dD+eE+$&c." E. G. Fischer[1] writes a finite expression $y=ax+bx^2+cx^3+\ldots.+px^r$ and, in the case of an infinite series of positive terms, he ends with "+etc."

437. M. Stern[2] writes "$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$ etc. $=\frac{\pi}{4}$." Enno Heeren Dirksen[3] of Berlin indicates in 1829 by "+etc." that the sum of the terms of an infinite series is intended, but a few years later[4] he marks an infinite progression thus: "1, 2, 3, 4 in inf." Martin Ohm[5] says: "Jede unendliche Reihe (P) .... $a+bx+cx^2+dx^3$ .... in inf. kann durch das kombinatorische Aggregat $S[P_a \cdot x^a]$ ausgedrückt werden, wenn man nur statt a nach und nach 0, 1, 2, 3, 4, in inf. gesetzt denkt, und wenn P_a den Koeffizienten von x^a vorstellt." L. Seidel[6] writes "$f(x, n)\ f(x, 1)+f(x, 2)$ .... in inf."; A. N. Whitehead[7] in 1911 ends with "+etc. ...." A notation for infinite series now frequently used is to place dots before and after the nth term, as in[8] $a_0+a_1(1/x)+a_2(1/x^2)+\ldots.+a_n(1/x)^n+\ldots.$

There are recent examples of rather involved notations to indicate the number of each term in a finite series, like the following of O. Stolz and J. A. Gmeiner.[9]

$$\overset{\smile\!\!\!\!\!1}{\frac{1}{b}}+\overset{\smile\!\!\!\!\!2}{\frac{1}{b}}+\ldots.+\overset{\smile\!\!\!\!\!a}{\frac{1}{b}}=\frac{a}{b}\,.$$

[1] Ernst Gottfried Fischer, *Theorie der Dimensionszeichen* (Halle, 1794), p. 27, 54.

[2] M. Stern, *Theorie der Kettenbrüche* (Berlin, 1834), p. 34. Reprint from *Crelle's Journal*, Vol. X, XI.

[3] *Crelle's Journal* (Berlin, 1829), p. 170.

[4] *Abhandlungen der K. P. Akademie d. Wissenschaften* (Berlin, 1832), Th. I, p. 77–107.

[5] Martin Ohm, *Versuch eines vollkommen consequenten Systems der Mathematik*, Vol. II (Berlin, 1829), p. 264, 265.

[6] *Abhandlungen d. Math.-Phys. Classe d. k. Bayerischen Akademie der Wissenschaften*, Vol. V (München, 1850), p. 384.

[7] A. N. Whitehead, *An Introduction to Mathematics* (New York and London), p. 212.

[8] W. B. Ford, *Studies on Divergent Series and Summability* (New York, 1916), p. 28.

[9] O. Stolz und J. A. Gmeiner, *Theoretische Arithmetik* (Leipzig), Vol. I (2d. ed., 1911), p. 104, 105, 109, 114, 121.

438. The sign Σ for summation is due to L. Euler[1] (1755), who says, "summam indicabimus signo Σ." This symbol was used by Lagrange,[2] but otherwise received little attention during the eighteenth century. A widely used notation for the sum of a series is the capital letter S; it is given, for instance, in G. S. Klügel's *Wörterbuch*.[3] The Σ to express "sum" occurs in 1829 in Fourier's *Theory of Heat*,[4] published in 1822, and in C. G. J. Jacobi's[5] elliptic functions of 1829. Cauchy[6] used three indices m, n, r, as in $\sum_{m}^{n}{}_{r}\, fr$. Alfred Pringsheim[7] marks the sum of an infinite series thus, $\sum_{0}^{\infty}{}_{\nu}\, a_\nu$.

Jahnke[8] adopts in one place, as a substitute for Σa_i, the simpler sign $\tilde{a}$, which he borrows from geodesists, to designate the sum $a_1+a_2+a_3+\ldots\ldots$ Additional symbols for sum are given in § 410.

SIGNS IN THE THEORY OF COMBINATIONS

439. *Binomial formula.*—The binomial formula, as Newton wrote it in his letter to Oldenburg of June 13, 1676, took this form, $(P+PQ)^{\overline{\frac{m}{n}}} = P^{\overline{\frac{m}{n}}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\text{etc.}$,[9] where A stands for the first term $P^{\overline{\frac{m}{n}}}$, B for the second term, and so on. Wallis,[10] in his *Algebra* of 1693, gives Newton's form of 1676 for the binomial formula. Leib-

[1] L. Euler, *Institutiones calculi differentialis* (St. Petersburg, 1755), Cap. I, § 26, p. 27.

[2] J. Lagrange, *Œuvres*, Vol. III, p. 451.

[3] G. S. Klügel's *Mathematisches Wörterbuch*, completed by J. A. Grunert, Part V (Leipzig, 1831), "Umformung der Reihen," p. 348; Part IV (1823), "Summirbare Reihe," p. 577.

[4] Joseph Fourier, *La théorie analytique de la chaleur* (Paris, 1822; Eng. trans. by A. Freeman, 1878), chap. iii, sec. 6, p. 208, and other places.

[5] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum* (1829); *Gesammelte Werke*, Vol. I (Berlin, 1881), p. 94.

[6] See G. Peano, *Formulaire mathématique*, Vol. IV (Turin, 1903), p. 132.

[7] See, for instance, *Encyklopädie der Math. Wissensch.*, Vol. I, Part I (Leipzig, 1898–1904), p. 77.

[8] E. Jahnke, *Archiv der Mathematik und Physik* (3d ser.), Vol. XXV (1916), p. 317.

[9] J. Collins, *Commercium Epistolicum* (London, 1712; ed. J. B. Biot and F. Lefort, Paris, 1856), p. 103.

[10] John Wallis, *De algebra tractatus* (1693), p. 376.

niz[1] wrote in 1695, "$\overline{m\,y+a}|^{(5)}=y^m+\frac{m}{1}y^{\overline{m-1}}\cdot a^1+\frac{m\cdot\overline{m-1}}{1\cdot 2}y^{\overline{m-2}}\cdot a^2$, etc.," where the (5) simply marks the equation as being equation No. 5. In Newton's *Quadratura curvarum*,[2] 1704, occurs the following passage: "Quo tempore quantitas x fluendo evadit $x+0$, quantitas x^n evadet $\overline{x+0}|^n$; id est per methodum serierum infinitarum, $x^n+n0x^{n-1}+\frac{nn-n}{2}00x^{n-2}+$ etc." In 1706 William Jones[3] gives the form

$$\overline{a+x}|^n=a^n+\frac{n-0}{1}a^{n-1}x+\frac{n-0}{1}\times\frac{n-1}{2}a^{n-2}x^2$$
$$+\frac{n-0}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}x^3+\text{etc.},$$

but Jones proceeds to develop the form given by Newton, except that Jones writes $a+ag$ where Newton has $P+PQ$. In 1747 Jones[4] adopted the abbreviations "$n'=n\cdot\frac{n-1}{2}$; $n''=n'\cdot\frac{n-2}{3}$; $n'''=n''\cdot\frac{n-3}{4}$; $n^{\text{iv}}=n'''\cdot\frac{n-4}{5}$." Another notation due to Kramp will be noted further on (§ 445). Cotes[5] wrote "$\overline{1+\mathcal{Q}}|^{\frac{m}{n}}=1+\frac{m}{n}\mathcal{Q}+\frac{m}{n}\times\frac{m-n}{2n}\mathcal{Q}\mathcal{Q}+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\mathcal{Q}^3+$ etc." Euler designated the binomial coefficient $\frac{n(n-1)\ldots\ldots(n-p+1)}{1\cdot2\cdot3\ldots\ldots p}$ by $\left(\frac{n}{p}\right)$ in a paper written in 1778 but not published until 1806,[6] and by $\left[\frac{m}{p}\right]$ in a paper of 1781, published in 1784.[7] Rothe[8] in 1820 denoted the pth binomial coefficient in the

[1] *Leibnizens Mathematische Schriften* (ed. C. I. Gerhardt), Vol. V (1858), p. 323.

[2] Isaac Newton, *Opera* (ed. S. Horsley), Vol. I (London, 1779), p. 336.

[3] William Jones, *Synopsis Palmariorum Matheseos* (London, 1706), p. 169, 170; the coefficients taking the form used by John Wallis in his *Algebra* of 1693, p. 358.

[4] W. Jones, *Philosophical Transactions for the Year 1747* (London, 1748), p. 563; abridged edition by John Martin, Vol. X (London, 1756), p. 17.

[5] Roger Cotes, *Harmonia mensurarum*, the tract *De methodo differentiali Newtoniana* (Cambridge, 1722), p. 30.

[6] L. Euler in *Nova acta acad. Petrop.*, Vol. XV (1806), *Math.*, p. 33

[7] L. Euler in *Acta Acad. Petrop.*, Vol. V (1784), pars prior, § 18, p. 89. See Cantor, *op. cit.*, Vol. IV, p. 206.

[8] H. A. Rothe, *Theorie der kombinatorischen Integrale* (Nürnberg, 1820). Taken from J. Tropfke, *op. cit.*, Vol. VI (2d ed., 1924), p. 44.

expansion of $(a+b)^n$ by n , and Ohm[1] in 1829 denoted the nth coefficient in $(a+b)^m$ by m_n. The notation $\binom{m}{p}$ which has become the more common was introduced in 1827 by von Ettingshausen,[2] and was used in 1851 by Raabe.[3] Stolz and Gmeiner[4] employ for binomial coefficients $\binom{m}{r}$ and also m_r, thus following the symbolism of Rothe and Ohm. Saalschütz[5] lets $(k)_h$ stand for $\binom{k}{h}$.

The quantic $ax^3+3bx^2y+3cxy^2+dy^3$, in which the terms are affected with the binomial coefficients arising in the expansion of $(x+y)^3$, is denoted by Cayley[6] by the abbreviation $(a, b, c, d \between x, y)$. When the terms are not affected by bionomial coefficients, Cayley put the arrow-head on the parenthesis, writing for instance $(a, b, c, d \between x, y)$ to denote $ax^3+bx^2y+cxy^2+dy^3$. Faà de Bruno[7] designated by $((x^p))F$ the coefficient of x^p in the development of F, which is a function of x.

An imitation of Newton's original mode of writing the binomial formula is seen in Stirling's notation for series: "I designate the initial terms of a series by the initial letters of the alphabet, A, B, C, D, etc. A is the first, B the second, C the third, and *sic porro*. I denote any term of this kind by the letter T, and the remaining terms in their order of succession by the same letter affixed with the Roman numerals I, II, III, IV, V, VI, VII, etc., for the sake of distinction. Thus, if T is the tenth, then T' is the eleventh, T'' the twelfth, T''' the thirteenth, and so on. And whichever term of this kind is defined as T, the ones that follow are generally defined by T' T'' T''' T^{iv}, etc. The distance of the term T from any given term I denote by the indeterminate quantity $\mathfrak{z}$."[8] Thus, following the notation of Newton ("more Newtoniana"), Stirling puts, in the series 1, $\frac{1}{2}x$, $\frac{3}{8}x^2$, $\frac{5}{16}x^3$, etc.,

$$A=1,\ B=\tfrac{1}{2}Ax,\ C=\tfrac{3}{4}Bx,\ \ldots\ldots,\ T'=\frac{\mathfrak{z}+\frac{1}{2}}{\mathfrak{z}+1}\,Tx.$$

[1] Martin Ohm, *op. cit.*, Vol. II (1829), p. 79.

[2] Andreas von Ettingshausen, *Vorlesungen über höhere Mathematik*, Vol. I (Vienna, 1827), p. 38. See E. Netto in Cantor, *op. cit.*, Vol. IV, p. 206; J. Tropfke, *op. cit.*, Vol. VI (1924), p. 44.

[3] J. L. Raabe in *Journal f. reine u. angewandte Mathematik*, Vol. XLII (1851), p. 350. See *Encyclopédie des sciences Math.*, Tome I, Vol. I (1904), p. 67, n. 20, 21.

[4] O. Stolz and J. A. Gmeiner, *Theoretische Arithmetik*, Vol. I (Leipzig, 1902), p. 187.

[5] Louis Saalschütz, *Bernoullische Zahlen* (Berlin, 1893), p. 3.

[6] See G. Salmon, *Modern Higher Algebra* (3d ed.; Dublin, 1876), p. 92.

[7] Faà de Bruno, *Théorie des formes binaires* (Turin, 1876), p. 161.

[8] James Stirling, *Methodus differentialis* (London, 1730), p. 3.

440. *Product of terms in an arithmetical progression.*—Machin,[1] in a paper on Kepler's problem, adopts the notation $\frac{\overset{a}{\smile}}{n+a}\Big(^m$ which he says, "denotes by its Index m on the Right-hand, that it is a Composite Quantity, consisting of so many Factors as there are Units in the Number m; and the Index a above, on the Left, denotes the common Difference of the Factors, decreasing in an Arithmetical Progression, if it be positive; or increasing, if it be negative; and so signifies, in the common Notation, the composite Number or Quantity, $\overline{n+a}\cdot\overline{n+a-a}\cdot\overline{n+a-2a}\cdot\overline{n+a-3a}\cdot$ and so on." Further on we shall encounter other notations for such a product, for instance, those of Kramp, Ampère, and Jarrett (§§ 445–47).

441. *Vandermonde's symbols.*—In 1770 C. A. Vandermonde, in an article on the resolution of equations,[2] adopted the following contractions:

"(A) pour $a+b+c+\&c$

(A^2) pour $a^2+b^2+c^2+\&c$

(AB) pour $ab+ac+\&+bc+\&c+$

(A^2B) pour $a^2b+b^2a+c^2a+\&c+a^2c+b^2c+c^2b+\&c+\&c+$.

Et en géneral par $(A^\alpha B^\beta C^\gamma \ldots .)r'$ indiquerai la somme de tous les termes différens qui résuteroient de celui-là, au moyen de toutes les substitutions possibles des petites lettres a, b, c, d, e, &c dans un ordre quel conque aux grandes, A, B, C, &c." He writes also $(A^\alpha B^\beta C^\gamma D^\delta E^\epsilon \ldots .)=\{\alpha\beta\gamma\delta\epsilon \ldots .\}$, or if several Greek letters are equal, he writes $\{\alpha^n\beta^p\gamma^q \ldots .\}$.

442. Laplace[3] represented the resultant of three equations by the symbolism $({}^1a^2\cdot b^3\cdot c)$, the indices being placed to the left of a letter and above.

443. *Combinatorial school of Hindenburg.*—The notation used by members of the combinatorial school in Germany is often so prolix and involved that a complete account of their symbolism transcends the limits of our space. The generalization of the binomial theorem so as to involve any power of any polynomial, be the number of terms

[1] John Machin, *Philosophical Transactions* (London), No. 147, p. 205 (Jan., etc., 1738); abridged by John Martyn, Vol. VIII, Part I, p. 78.

[2] C. A. Vandermonde in *Histoire de l'académie r. d. sciences*, année 1771 (Paris, 1774), p. 370, 371.

[3] P. S. Laplace, *Histoire de l'académie, r. d. sciences* (Paris), année 1772, p. 267, 294, 2. partie; Laplace, *Œuvres*, Vol. VIII, p. 365–406. See also Th. Muir, *The Theory of Determinants in Historical Order of Development*, Part I (2d ed., 1906), p. 30.

in the polynomial finite or infinite, was one of the problems considered. Carl Friedrich Hindenburg, in a treatise of 1778,[1] represents the binomial coefficients by capital German letters, thus

$$^{m}\mathfrak{A}=\frac{m}{1}\,,\qquad {}^{m}\mathfrak{B}=\frac{m(m-1)}{1\cdot 2}\,,\qquad {}^{m}\mathfrak{C}=\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\,,\ldots .$$

This notation is followed by G. S. Klügel in his *Wörterbuch*.[2] Netto remarks that Hindenburg adheres to the practice which had been at least in part abandoned by Leibniz as inconvenient—the practice of using the alphabetical arrangement of the letters for the designation of order and for enumeration. Hindenburg employs the upright Latin capitals A, B, C to mark combinations, the slanting Latin capitals *A*, *B*, *C* for permutations. A superscript stroke placed to the left of a letter, ′A ′*A*, means a combination or permutation *without* repetitions, while a stroke placed to the right, A′ *A*′, means *with* repetitions. To mark coefficients that are not binomial, but polynomial, he uses *small* German letters in place of the *capital* German letters given above.

In a book of 1779, Hindenburg uses small and capital letters in Roman, Greek, and German type. He employs superscripts placed not only before or after a letter, but also above the letter, the superscripts either inclosed in a parenthesis or not. He[3] calls ^{m}b, ^{m}c, ^{m}d, *signa indefinita*, used in the manner illustrated in the following expressions:

$$A^m=a^m$$

$$B^m=(A+b)^m=A^m+mA^{m-1}b+\frac{m\cdot m-1}{1\cdot 2}A^{m-2}b^2+\&c=a^m+{}^{m}b\,.$$

$$C^m=(B+c)^m=\,\ldots\ldots=B^m+{}^{m}c=a^m+{}^{m}b+{}^{m}c\,.$$

Further on (p. 17) Hindenburg puts

$$a[a+b+c+d\ldots\ldots+\omega]={}'A\,,$$
$$a['A+'B+'C+'D\ldots\ldots+'\Omega]={}''A\,,$$
$$a[''A+''B+''C+''D\ldots\ldots+''\Omega]={}'''A\,,\ldots\ldots,$$
$$d[''D\ldots\ldots+''\Omega]={}'''D,\ \text{etc.},$$
$$^{n}D=d[^{n-1}D\ldots\ldots+{}^{n-1}\Omega]\,.$$

[1] C. F. Hindenburg, *Methodus nova et facilis serierum infinitarum*, etc. (Göttingen, 1778). Our knowledge of this publication is drawn from E. Netto in Cantor, *op. cit.*, Vol. IV, p. 205–7.

[2] G. S. Klügel, *Mathematisches Wörterbuch*, 1. Theil (Leipzig, 1803), art "Binomial—Coefficienten."

[3] C. F. Hindenburg, *Infinitinomii dignitatum .... Historia, leges ac formulae* (Göttingen, 1779), p. 5, 17, 18, 20.

He introduces (p. 21) $\int$ for *summam partium* (the parts chosen in a certain manner), which is to be distinguished from the sign of integration $\int$, but like $\int$ has *vim transitivam*, i.e., is distributive in addition. Later still (p. 159), writing $B=b+c+d+\&c$, $C=c+d+e+\&c, \ldots.$, he puts $\overset{2}{B}=bB+cC+dD+\&c$, $\overset{2}{C}=cC+dD+eE+\&c$, and so on; also (p. 161) $\overset{(2)}{B}=bC+cD+dE \ldots.$, $\overset{(2)}{C}=cD+dE \ldots.$, $\overset{(3)}{C}=c\overset{(2)}{D}+d\overset{(2)}{E} \ldots.$, and so on.

444. In 1796 and 1800 Hindenburg published collections of papers on combinatorial analysis. The first collection[1] opens with an article by J. N. Tetens of Copenhagen who lets $|n|$ stand for the coefficient of the nth term of a polynomial, also lets the coefficient of the nth term (*terminus generalis*) in the expansion of $(a+bx+cx^2+\ +|n|x^{n-1}+\ +)^m$ be indicated by $T(a+bx+\ +|n|x^{n-1})^m$ or simply by $T(a+\ +|n|)^m$. In a footnote Hindenburg compares the symbols of Tetens with symbols of his own; thus, $T(a+\ +|n|)^m=p^m\mathfrak{X}^n$, etc., where in Hindenburg's notation of that time, $p^m\mathfrak{X}^n$ is the nth coefficient of the power p^m.

In 1800 Hindenburg[2] refers to Heinrich Bürmann who advanced a purely combinatorial notation in his *Developpment des fonctions combinatoires*. Hindenburg states that Bürmann's short and expressive signs cannot be explained at this time, but states that the simplest *monogrammata* of Bürmann from which the others are formed by additions and changes are ⊓ for series, [for combinations,] for discerption. In 1803 Hindenburg and Bürmann brought out a joint publication, *Ueber combinatorishe Analysis*, and in 1807 Bürmann published at Mannheim a *Pangraphie*, or system of universal notation. We have not seen these two publications. As we point out elsewhere (§ 428), J. W. F. Herschel adopted a few of Bürmann's symbols.

445. *Kramp on combinatorial notations.*—Kramp[3] of Cologne in 1808 expressed himself on matters of notation as follows: "If one designates by Q any polynomial, ordered according to the power of the variable x, such that $ax^n+bx^{n+r}+cx^{n+2r}+$, etc., the notation $Q\mathfrak{k}_1$, $Q\mathfrak{k}_2$, $Q\mathfrak{k}_3$, etc., will be very convenient for designating the coefficients of that polynomial, namely, a, b, c, etc. The coefficient $Q\mathfrak{k}\ m$ is accordingly the one preceding the power of x whose exponent is $n+(m-1)r$.

[1] *Sammlung combinatorisch-analytischer Abhandlungen*, herausgeg. von Carl Friedrich Hindenburg, Erste Sammlung (Leipzig, 1796), p. 6, 7.

[2] *Op. cit.*, Zweyte Sammlung (1800), p. xiii.

[3] Christian Kramp, *Élémens d'arithmétique universelle* (Cologne, 1808), "Notations."

The polynomial Q is therefore $Q\backprime\mathfrak{k}1x^n+Q\backprime\mathfrak{k}2\cdot x^{n+r}+Q\backprime\mathfrak{k}3\cdot x^{n+2r}+$, etc. The power of the polynomial Q^h, according to this same notation, becomes $Q^h\backprime\mathfrak{k}1\cdot x^{nh}+Q^h\backprime\mathfrak{k}2\cdot x^{nh+r}+Q^h\backprime\mathfrak{k}3\cdot x^{nh+2r}+$, etc. And if by S one understands any other polynomial, ordered according to the powers of that same variable, such that $S\backprime\mathfrak{k}1\cdot x^l+S\backprime\mathfrak{k}2\cdot x^{l+r}+S\backprime\mathfrak{k}3\cdot x^{l+2r}+$, etc., the product of these two polynomials Q^hS will be identical with $(Q^hS)\backprime\mathfrak{k}\cdot x^{nh+l}+(Q^hS)\backprime\mathfrak{k}2x^{nh+l+r}+(Q^hS)\backprime\mathfrak{k}3\cdot x^{nh+l+2r}+$, etc.

"Professor Hindenburg appears to be the first mathematician who has felt the indispensable need of this *local* notation in the present state of Analysis; it is moreover generally adopted today by the mathematicians of his nation. Above all the polynomial coefficients $Q^h\backprime\mathfrak{k}_1$, $Q^h\backprime\mathfrak{k}_2$, $Q^h\backprime\mathfrak{k}_3$, etc. recur without ceasing in all that vast part of Analysis which has for its object the development in series of any function whether explicit or implicit. . . . .

"In the eighteenth chapter and in most of those which follow, I have conveniently employed the German letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc., to denote the binomial factors of the power to the exponent n. One has therefore, $\mathfrak{a}=n$; $\mathfrak{b}=\dfrac{n(n-1)}{1\cdot2}$ and so on.

"I employ the Latin letter D, placed before the polynomial function Q, and I have named *first derivative* (*première dérivée*), what results from that function when one multiplies all its terms by their respective exponents and one then divides them by x. In the case that $Q=a+bx+cx^2+dx^3+$etc. one will therefore have, $DQ=b+2cx+3dx^2+4ex+{}^3$etc. One has in the same way for a *second derivative* (*seconde dérivée*), $D^2Q=2c+6dx+12ex^2+20fx^3+$etc. . . . .

"The capital German letters are in general signs of functions. I employ particularly to this end the letters $\mathfrak{F}$ and $\mathfrak{P}$. Thus $\mathfrak{F}(x)$ designates any function whatever of the variable; $\mathfrak{P}(x)$ designates another.

"The derivatives (*dérivées*) of Arbogast designate therefore the simple coefficients of the series; mine are veritable polynomial functions, developed in series, of which only the first terms are the derivatives of Arbogast. The notation of these last is identical with the local notation $\mathfrak{F}(a+x)\backprime\mathfrak{k}1$, $\mathfrak{F}(a+x)\backprime\mathfrak{k}2$, $\mathfrak{F}(a+x)\backprime\mathfrak{k}3$ etc. for the first derivatives; and for the second, with $(\mathfrak{F}X)\backprime\mathfrak{k}1$, $(\mathfrak{F}X)\backprime\mathfrak{k}2$, $(\mathfrak{F}X)\backprime\mathfrak{k}3$ etc. always in the case that the exponents of the powers of x in the function X are the terms of the progression of the natural numbers 0, 1, 2, 3, etc. . . . .

"My researches on the calculus of derivatives go back to the year 1795. They appear for the first time in the work published in 1796

by Professor Hindenburg, under the title *Der polynomische Lehrsatz*, where this mathematician did me the honor of joining my essay on combinatorial analysis with his and also with those of Messrs. Tetens, Pfaff and Klügel. . . . .

"For the designation of the product whose factors form among themselves an arithmetical progression, such as $a(a+r)(a+2r) \ldots . (a+nr-r)$, I have retained the notation $a^{n|r}$ already proposed in my *analyse des refractions;* I have given it the name *facultés.* Arbogast substituted for it the choicer and more French designation of *factorielles.*"

Klügel[1] uses Kramp's notation $a^{m,r}$ for $a(a+r) \ldots . (a+mr-r)$ and calls it a *Facultät.*

446. *Signs of Argand and Ampère.*—Independently of German writers, a few symbols were introduced by French writers. J. R. Argand,[2] in treating a "Problème de combinaisons," designates by (m, n) the "ensemble de toutes les manières de faire avec m choses n parts ... et par $Z(m, n)$ le *nombre* de ces manières." A few years later A. M. Ampère[3] lets $\overset{1}{[x]}=x$, $\overset{2}{[x]}=\overset{1}{[x]}(x+p)$, $\overset{3}{[x]}=\overset{2}{[x]}(x+2p) \ldots .$, $\overset{m+1}{[x]}=\overset{m}{[x]}(p+mp)$. This notation resembles that of Vandermonde; Ampère refers to the work of Vandermonde and Kramp. Ampère's notation is used by Lentheric[4] of Montpellier.

Crelle[5] adopted the signs when m is an integer: $(u, +x)^m = u(u+x)(u+2x)(u+3x) \ldots . (u+[m-1]x)$, $(u,+x)^{-y}=\frac{1}{(u-yx, +x)^y}$. Schellbach[6] added to these symbols the following:

$$a_0^n,+k=a_0\, a_k\, a_{2k}\, a_{3k} \ldots . a_{nk-k}\,,$$
$$f^n(x,+y)=f(x)\, f(x+y)\, f(x+2y) \ldots . f(x+ny-y),$$
$$(1-a_n,+1)^{-n}=\frac{1}{(1-a_0)(1-a_1)(1-a_2) \ldots . (1-a_{n-1})}\,.$$

Schellbach gives the symbolism $n!a_\delta$ to mark the occurrence of n quantities $a_0, a_1, a_2, \ldots . , a_{n-1}$, where δ takes successively the values

[1] G. S. Klügel, *Mathematisches Wörterbuch*, 1. Theil (Leipzig, 1803), art. "Facultät."

[2] J. R. Argand in J. D. Gergonne's *Annales de mathématiques pures et appliquées* (Nismes), Vol. VI (1815 and 1816), p. 21.

[3] *Op. cit.*, Vol. XV (1824–25), p. 370.

[4] *Op. cit.*, Vol. XVI (1825–26), p. 120.

[5] A. L. Crelle, in *Crelle's Journal*, Vol. VII (1831), p. 270, 271.

[6] Karl Heinrich Schellbach, *Crelle's Journal*, Vol. XII (1834), p. 74, 75. Schellbach gave a discussion of mathematical notation in this article.

0, 1, ... , $n-1$. Accordingly, $f(n!x_\delta)$ stands for a function of x_0, x_1, ... , x_{n-1}. He writes $(m, n!\,a_\delta)$ = the combinations, without repetition, of the elements $a_0, a_1, \ldots, a_{n-1}$, taken m at a time. And $[m, n!, a_\delta]$ = the combinations with repetitions. He lets also $n|a_\sigma = a_0 + a_1 + \ldots + a_{n-1}$, where σ takes successively the values 0, 1, ... , $n-1$. Also[1] $(a+b)^3 = 4 \mid [3-\sigma \downarrow 1+\sigma!\, a+1+\delta]\, [\sigma \uparrow 4-\sigma!\, \delta-a]\left(\frac{b-\sigma,\ +1}{1,\ +1}\right)^3$, and similarly for $(a+b)^n$, where the arrows indicate (p. 154) the direction of the combination.

447. *Thomas Jarrett.*—An extensive study of algebraic notations was made by Thomas Jarrett (1805–82), of Catharine Hall, at Cambridge in England. He[2] published an article in 1830, but gave a much fuller treatment of this subject in an essay[3] of 1831. He remarks that the demonstration of the legitimacy of the separation of the symbols of operation and quantity, with certain limitations, belongs to Servois. Jarrett refers to Arbogast, J. F. W. Herschel, Hindenburg, Lacroix, Laplace, Schweins, and Wronski. Jarrett points out that the following notations used by him are not original with him: First, $E_x\phi(x)$ for $\phi(x+Dx)$ is partly due to Arbogast; second, $d_x^n u$ for $\frac{d^n u}{dx^n}$ is due to Lacroix, although not used by him, being merely pointed out in a single line (*Calcul Diff.*, Vol. II, p. 527); third, $(u)_{x=a}$ for the value assumed by u, when x is put equal to a, belongs to Schweins.

The principal symbols introduced by Jarrett are as follows:[4]

$\overset{n}{S}_m a_m$, the sum of n terms, of which the mth is a_m (p. 1).

$\overset{n;r}{S}_m a_m$, the rth term must be omitted.

$\underset{m}{\overset{r}{S}}\,\underset{n}{\overset{s}{S}}\, a_{m,n}$, the sum of r terms of which the mth is $\underset{m}{\overset{s}{S}}\, a_{m,n}$ (p. 5).

$\overset{m,n}{S}_{r,+s}({}^s a_r)$, the sum of every term that can be formed with the following conditions: each term in the product of m quantities in which r has the values of the successive

[1] Schellbach, *op. cit.*, p. 154, 156.

[2] Thomas Jarret in *Transactions of the Cambridge Philosophical Society*, Vol. III (Cambridge, 1830), p. 67.

[3] *An Essay on Algebraic Development containing the Principal Expansions in Common Algebra, in the Differential and Integral Calculus and in the Calculus of finite Differences. . . .* By the Rev. Thomas Jarrett, M.A., Fellow of Catharine Hall, and Professor of Arabic in the University of Cambridge (1831).

[4] Thomas Jarrett, *op. cit.* (1831), "Index to the Symbols."

natural numbers, while s has any m values such that their sum shall be n, zero being admissible as a value of s, and repetitions of the same value of that letter being allowed in the same term (p. 78).

$\varpi_a^m \phi^{(a)}$, coefficient of x^m in the development of $\phi(\overset{\infty}{S}_m a_{m-1} x^{m-1})$ (p. 79).

$\overset{n,r}{S}_m a_m$, the sum of the series formed by giving to m every integral value from n to r both inclusive; zero being also taken as a value if n is either zero or negative (p. 136).

$\overset{n}{P}_m a_m$, the product of n factors, of which the mth is a_m (p. 12).

$\overset{n,r}{P}_m a_m$, the rth factor must be omitted.

$\underline{|a}_{n,m} = a(a+m)(a+2m) \ldots . (a+\overline{n-1}\cdot m)$ (p. 15).

$\underline{|a}_{n} = a(a-1)(a-2) \ldots . (a-n+1)$.

$\underline{|a} = a(a-1)(a-2) \ldots . 2\cdot 1$.

$\overset{n}{\underset{m}{\{}} a_m + b_n \underset{m+1}{\{} \ldots . \underset{n+1}{\{} c \underset{n+1}{\}} \ldots .\}$ denotes the result of the combination of the symbols $\underset{1}{\{} a_1 + b_1 \underset{2}{\{} a_2 + b_2 \ldots . \underset{n}{\{} a_n + b_n \underset{n+1}{\{} c \underset{n+1}{\}} \ldots . \underset{1}{\}}$;

the brackets being omitted after the expansion, if they are then without signification (p. 19).

Theorem. $\overset{n}{\underset{m}{\{}} a_m + b_m \underset{m+1}{\{} \ldots . \underset{n+1}{\{} c = \overset{n}{S}_m a_m \cdot \overset{m-1}{P}_r b_r + c \cdot \underset{r}{\overset{m}{P}} b_r$.

$\overset{m,n}{C}_r a_r$, the sum of every possible combination (without repetition of any letter in the same combination) that can be formed by taking m at a time of n quantities of which the rth is a_r.

$\overset{m,n;s}{C}_r a_r$, a_s is to be everywhere omitted.

$\overset{\overline{m,n-m}}{C}_{r,s}(a_r \cdot b_s)$, n quantities of which the rth is a_r, and n others of which the sth is b_s, every possible combination being formed of the first series, by taking them m at a time, each combination thus formed being multiplied by $n-m$ quantities of the second series, so taken that in each of the combinations the whole of the natural numbers from 1 to n shall appear as indices: thus (p. 22).

$\overline{2,3}$
$C_{r,s}(a_r \cdot b_s) = a_1a_2b_3b_4b_5 + a_1a_3b_2b_4b_5 + a_1a_4b_2b_3b_5 + a_1a_5b_2b_3b_4 + a_2a_3b_1b_4b_5 + a_2a_4b_1b_3b_5 + a_2a_5b_1b_3b_4 + a_3a_4b_1b_2b_5 + a_3a_5b_1b_2b_4 + a_4a_5b_1b_2b_3$.

$(\phi+\psi)_n u$ means $\{(\phi+\psi)(\phi+\psi) \ldots . (\phi+\psi)\}u$, ($n$ round parentheses) (p. 41).

$\overset{n}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\{}} u$ means $\{(\phi_1+\psi_1)(\phi_2+\psi_2) \ldots . (\phi_n+\psi_n)\}u$.

$(u)_{x=a}, \phi_{x=a}(u)$, denote respectively the values of u, and $\phi_x(u)$, when x is put equal to a; this substitution, in the latter case, not being made until after the operation indicated by ϕ_x has been performed (p. 45).

$E_x^{(u)}$ means that in u, any function of x, $x+h$ is substituted for x.

$D_x^{(u)}$ means the excess of the new value of u above the original value.

$E_{x,y}^{(u)}$ expresses either $E_x \cdot E_y(u)$ or $E_y \cdot E_x(u)$ (p. 58).

$\mathcal{C}_{2m-1} = \overset{2m}{\underset{+r}{A}}\left(\frac{-1}{\underline{|r+1}}\right)$, where the right member is called the $(2m-1)$th number of Bernoulli; $\mathcal{C}_1 = \frac{1}{2\underline{|3}} = ,083$ (p. 89).

A turn through 180° of Jarrett's sign for n-factorial yields $\urcorner$, a symbol introduced by Milne[1] in the treatment of annuities. He lets a stand for the expectation of life by an individual A, ${}_ta$ the probability of his surviving t years, and $\urcorner_t a$ the expectation of life after the expiration of t years. This sign was used similarly by Jones[2] who lets $a_{(m)}\urcorner_n$ be the present value of £1 per annum, to be entered upon after n years, m being the present age.

448. *Factorial "n."*—The frequency of the occurrence of n-factorial in algebra and general analysis gives this expression sufficient importance to justify a separate treatment, even at the risk of some slight repetition of statements. In 1751 Euler[3] represented the product 1.2.3. . . . . m by the capital letter M. "Ce nombre de cas 1.2.3.4. . . . . m etant posé pour abréger $=M$. ..." Probably this was not intended as a general representation of such products, but was introduced simply as a temporary expedient. The very special relation

[1] Joshua Milne, *Annuities and Assurances*, Vol. I (London, 1815), p. 57, 58.

[2] David Jones, *Value of Annuities*, Vol. I (London, 1843), p. 209.

[3] L. Euler, "Calcul de la Probabilité dans le jeu de Recontre," *Histoire de l'académie r. d. sciences et des belles lettres de Berlin*, année 1751 (Berlin, 1753), p. 259, 265.

m to M would go against the use of M as the product of, for example, $1.2.3.\ldots.r$, or of 1.2.3.4.5.6. A little-known suggestion came in 1774 from J. B. Basedow[1] who used a star, thus $5^* = 5.4.3.2.1$. Other abbreviations were used in 1772 by A. T. Vandermonde: "Je représente par $\overset{n}{[p]}$ le produit de n facteurs ... $p\cdot(p-1)\cdot(p-2)\cdot(p-3)$ ... ou le produit de n termes consécutifs d'une suite dont les premières différences sont 1, et les secondes différences sont zéro."[2] He writes $\overset{n}{[p]} = p\cdot(p-1)\cdot(p-2)\cdot(p-3)\ \ldots.\ (p-n+1)$; he finds $\overset{0}{[p]} = 1$, $\overset{-n}{[p]} = \frac{1}{(p+1)\cdot(p+2)\cdot(p+3)\cdot(p+4)\ \ldots.\ (p+n)}$, $2^4 \overset{4}{\left[\frac{11}{2}\right]} = 11.9.7.5.$, $\overset{-5}{[0]} = \frac{1}{1.2.3.4.5.}$, $\frac{1}{2}\pi = \overset{\frac{1}{2}}{[\tfrac{1}{2}]}\overset{-\frac{1}{2}}{[-\tfrac{1}{2}]} = \frac{2.2.4.4.6.6.\ldots.}{1.3.3.5.5.7.\ldots.}$. The special case when $p=n=$ a positive integer would yield the product $n(n-1)\ \ldots.\ 3.2.1.$, but Vandermonde was operating with expressions in form more general than this (see also § 441).

A sign for n-factorial arises as a special case of a more general notation also in Christian Kramp of Strasbourg who in his *Élémens d'arithmétique universelle* (1808) and in special articles[3] lets $a^{m|r}$ stand for $a(a+r)(a+2r)\ \ldots.\ [a+(m-1)r]$, and uses the special forms $a^{m|0} = a^m$, $a^{-m|-r} = \frac{1}{(a+r)^{m|r}}$, $1^{m|1} = 1.2.3\ \ldots.\ m$ "ou a cette autre forme plus simple $m!$" In 1808 Kramp said: "Je me sers de la notation trés simple $n!$ pour désigner le produit de nombres décroissans depuis n jusqu'à l'unité, savoir $n(n-1)(n-2)\ \ldots.\ 3.2.1.$ L'emploi continuel de l'analyse combinatoire que je fais dans la plupart de mes démonstrations, a rendu cette notation indispensable."[4] In a footnote to Kramp's article, the editor, J. S. Gergonne, compares the notations of Vandermonde and Kramp. "Vandermonde fait $a\cdot(a-1)\cdot(a-2)\ \ldots.\ (a-m+1) = [a]^m$, d'où il suit qu'en rapprochant les deux notations, ou a $[a]^m = (a-m+1)^{m|1} = a^{m|-1}, \ldots.,\ 1.2.3.4\ \ldots.\ m = 1^{m|1} = m!$" Kramp's notation $1^{m|1}$ found its way into Portugal where Stockler[5] used it in 1824.

[1] Johann Bernhard Basedow, *Bewiesene Grundsätze der reinen Mathematik*, Vol. I (Leipzig, 1774), p. 259.

[2] A. T. Vandermonde, *Histoire de l'académie r. d. sciences*, année 1772, Part I (Paris, 1775), *Mém.*, p. 490, 491.

[3] J. D. Gergonne, *Annales de Mathématiques*, Vol. III (1812 et 1813), p. 1.

[4] C. Kramp, *Élémens d'arithmétique universelle* (Cologne, 1808), "Notations." See also p. 219.

[5] Francisco de Borja Garção Stockler, *Methodo inverso dos Limites* (Lisbon, 1824), p. 35.

A new designation for n-factorial was introduced by Legendre. In 1808 he wrote in an article, "Dans la première qui traite des intégrales de la forme $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$, prises depuis $x=0$ jusqu'à $x=1$ désignées par Euler par le symbole $\left(\frac{p}{q}\right)$, on peut regarder comme choses nouvelles."[1] Legendre designated the integral by the capital Greek letter gamma, Γ; and continued this usage in his integral calculus[2] of 1811. As the value of this definite integral is n-factorial, the symbol $\Gamma(n+1)$ came to stand for that value.

449. Occasionally the small letter pi was used to represent n-factorial, as, for instance, in Ruffini's theory of equations[3] of 1799, where one finds "$\pi=1.2.3.4. \ldots . m$," or "un' Equazione del grado $1.2.3. \ldots . m$, che chiamo $\pi. \ldots .$" More frequent is the use of the capital letter Π; thus Gauss[4] employed for n-factorial the notation $\Pi(n)$; Jacobi[5] Π_4 for 4! H. Weber[6] in 1893 wrote $\Pi(m)=1.2.3. \ldots m$. There are a few obsolete notations; for instance, Henry Warburton,[7] of Cambridge, England, in 1847, represents n-factorial by $1^{n|1}$, as a special case of $S^{n|1}\equiv s(s+1).(s+2) \ldots . (s+[n-1])$, which resembles Kramp's $a^{m|r}$.

De Morgan[8] (who ordinarily uses no contracted symbol) employed in 1838 the designation $[n]$, as a special case, apparently, of Vandermonde's[9] $[p]^m\equiv p(p-1)(p-2) \ldots . (p-[m-1])$. Carmichael[10] in 1855 and 1857, when hard pressed for space in writing formulas, uses the

[1] *Mémoires de l'institut national, sciences math. et phys.*, année 1808 (Paris, 1809), p. 14, 15.

[2] A. M. Legendre, *Exercices de calcul intégral*, Vol. I (Paris, 1811), p. 277.

[3] Paolo Ruffini, *Teoria generale delle equazioni* (Bologna, 1799), p. 97, 244, 254.

[4] C. F. Gauss, *Commentationes Societatis regiae scientiarum Gottingensis recentiores*, Vol. II (1811–13), *Math. mém.*, nº 1, p. 26; *Werke*, Vol. III (Göttingen, 1866), p. 146.

[5] C. G. J. Jacobi, *Crelle's Journal*, Vol. XII (1834), p. 264.

[6] H. Weber, *Mathematische Annalen*, Vol. XLIII (1893), p. 535.

[7] H. Warburton, *Cambridge Philosophical Transactions*, Vol. VIII (1849), p. 477.

[8] A. de Morgan, *Essay on Probabilities, Cabinet Cyclopoedia*, p. 15 (London, 1838).

[9] A. T. Vandermonde in *Mémoires de l'Académie des Sciences* (1772), première partie, p. 490.

[10] Robert Carmichael, *Treatise on the Calculus of Operations* (London, 1855); German edition, *Operations Calcul* (Braunschweig, 1857), p. 30–55.

designation $\bar{n}$. Weierstrass[1] in 1841 and later Schlömilch[2] chose the sign m' for our $m!$.

Relating to the origin of the notation $\underline{|n}$ for "n-factorial," nothing has been given in histories, except the statement that it has been in use in England. The notation $\underline{|n}$ was suggested in 1827 by Jarrett who had just graduated from St. Catherine's College in Cambridge, England, with the degree of B.A. It occurs in a paper "On Algebraic Notation" that was printed in 1830[3] (see § 447). The passage in question reads as follows: "A factorial of this kind consisting of m factors, of which n is the first, and of which the common difference is $\pm r$, may be denoted by $\dfrac{\underline{|\ n}}{m\pm r}$; the particular case in which the common difference is -1, we may represent by $\dfrac{\underline{|n}}{m}$ and if, in this case, $m\equiv n$, the index subscript may be omitted:
Thus

$$\frac{\underline{|\ n}}{m\pm r}=n(n\pm r)(n\pm 2r)\ \ldots .\ (n\pm\overline{m-1}r)$$

$$\frac{\underline{|n}}{m}=n(n-1)(n-r)\ \ldots .\ (n-\overline{m-1})$$

$$\underline{|n}=n(n-1)(n-2)\ \ldots .\ 1\ .\text{"}$$

For a quarter of a century the notation $\underline{|n}$ was neglected. In 1846, Rev. Harvey Goodwin used it freely in an article, "On the Geometrical Representation of Roots of Algebraic Equations," that was printed in 1849.[4] In 1847 Goodwin published his *Elementary Course in Mathematics*, a popular educational manual which reached several editions, but, to our surprise, he did not make use of any contracted notation for factorial n in this text. In fact, the symbol $\underline{|n}$ made no substantial headway in England until it was adopted by Todhunter about 1860, and was used in his popular texts. In his *History of Probability*[5] he says: "I have used no symbols which are not common to all mathematical literature, except $\underline{|n}$ which is an abbreviation for the product 1.2. . . . . n, frequently used but not

[1] K. Weierstrass, *Mathematische Werke*, Vol. I (Berlin, 1894), p. 14, 50.

[2] O. Schlömilch, *Zeitschrift für Math. u. Physik*, Vol. II (1857), p. 139.

[3] *Transactions of the Cambridge Philosophical Society*, Vol. III, p. 67 (Cambridge, 1830).

[4] *Cambridge Philosophical Transactions*, Vol. VIII (1849), p. 343.

[5] Isaac Todhunter, *History of the Mathematical Theory of Probability*, p. viii and lx (Cambridge and London, 1865).

universally employed; some such symbol is much required, and I do not know of any which is preferable to this, and I have accordingly introduced it in all my publications." In 1861, $\underline{|n}$ was used by Henry M. Jeffery.[1] Some use of it has been made in Sweden,[2] though less often than of $n!$.

In the United States $\underline{|n}$ was probably introduced through Todhunter's texts. In the first volume (1874) of J. E. Hendricks' *Analyst* (Des Moines, Iowa), both the notation $\underline{|n}$ and $n!$ are used by different writers. The latter notation, though simpler, was used in elementary texts of this country less frequently than the first. The notation $\underline{|n}$ was adopted by such prominent textbook writers as Joseph Ficklin (*Complete Algebra*, copyright 1874), Charles Davies (revised Bourdon's *Elements of Algebra* [1877]), Edward Olney (1881), and about the same time by George A. Wentworth, Webster Wells, E. A. Bowser, and others. Thus it became firmly rooted in this country.[3] Among the French and German authors $\underline{|n}$ has met with no favor whatever. In 1915 the Council of the London Mathematical Society,[4] in its "Suggestions for Notation and Printing," recommended the adoption of $n!$ for $\underline{|n}$, $(n!)^2$ for $(\underline{|n})^2$, $2^n \cdot n!$ for $2^n \underline{|n}$.

The notation $n!$ is used after Kramp in Gergonne's *Annales* by J. B. Durrande[5] in 1816, and by F. Sarrus[6] in 1819. Durrande remarks, "There is ground for surprise that a notation so simple and consequently so useful has not yet been universally adopted." It found wide adoption in Germany, where it is read "n-Fakultät."[7] Some texts in the English language suggest the reading "n-admiration"[8] (the exclamation point [!] being a note of admiration), but most texts prefer "factorial n," or "n-factorial." In Germany, Ohm,[9] whose books enjoyed popularity for many years, used the notation $n!$

[1] *Quarterly Journal of Mathematics*, Vol. IV, p. 364 (1861).

[2] *Encyclopédie des sciences mathématiques pures et appliquées*, Tome I, Vol. I, p. 65 (1904).

[3] In a few publications Jarrett's factorial symbol is given in the modified form $\underline{n|}$. See, for example, Thomas Craig's *Treatise on Linear Differential Equations*, Vol. I, p. 463 (New York, 1889), and Webster's *New International Dictionary of the English Language* (Springfield, 1919), under the word "Factorial."

[4] *Mathematical Gazette*, Vol. VIII (London, 1917), p. 172.

[5] Gergonne, *Annales de mathématiques*, Vol. VII (1816 and 1817), p. 334.

[6] *Op. cit.*, Vol. XII (1821 and 1822), p. 36.

[7] E. Pascal, *Repertorium*, Vol. I, p. 43 (Leipzig und Berlin, 1910).

[8] W. E. Byerly, *Elements of the Differential Calculus*, p. 120 (Boston, 1880).

[9] M. Ohm, *System der Mathematik*, Vol. II, p. 17 (Berlin, 1829).

about 1829. It was used in 1847 by Eisenstein, then Privat Docent in Berlin, in an article quoted by Sylvester[1] in 1857. Chrystal's *Algebra* (1889) came out for $n!$, though in the nineteenth century it was much less frequent in England than $\underline{|n}$.

In the United States $n!$ was used by W. P. G. Bartlett[2] as early as the year 1858, and by Simon Newcomb[3] in 1859. It was adopted mainly by a group of men who had studied at Harvard, Pliny Earl Chase[4] (later professor of physics at Haverford), James Edward Oliver[5] (later professor of mathematics at Cornell), and C. S. Peirce[6] the logician. Afterward $n!$ was used in Newcomb's *Algebra* (1884), I. Stringham's edition of Charles Smith's *Algebra* (1897), M. Merriam and R. S. Woodward's *Higher Mathematics* (1898), Oliver, Wait, and Jones's *Algebra* (1887), and in others.

In the present century the notation $n!$ has gained almost complete ascendancy over its rivals. It is far more convenient to the printer.

Remarkable is the fact that many writings, both advanced and elementary, do not use any contracted notation for n-factorial; the expanded notation $1, 2, 3 \ldots . n$ is adhered to. The facts are that a short mode of designation is not so imperative here as it is for "square root," "cube root," or "the nth power." We have seen that Harvey Goodwin of Caius College, Cambridge, made liberal use of $\underline{|n}$ in a research article, but avoided it in his *Elementary Course*. Instinctively he shrunk from the introduction of it in elementary instruction. We have here the issue relating to the early and profuse use of symbolism in mathematics: Is it desirable? In the case of n-factorial some writers of elementary books of recognized standing avoid it. More than this, it has been avoided by many writers in the field of advanced mathematics, such as J. J. Sylvester, A. Cayley, H. Laurent, E. Picard, E. Carvallo, E. Borel, G. B. Airy, G. Salmon, W. Chauvenet, Faà de Bruno, P. Appell, C. Jordan, J. Bertrand, W. Fiedler, A. Clebsch.

[1] G. Eisenstein, *Mathematische Abhandlungen* (Berlin, 1847); J. J. Sylvester, *Quarterly Journal of Mathematics*, Vol. I, p. 201 (London, 1857).

[2] J. D. Runkle's *Mathematical Monthly*, Vol. I, No. 3, p. 84–87 (Cambridge, Mass., 1858).

[3] J. D. Runkle, *op. cit.*, Vol. I (1859), p. 331, 396.

[4] *Trans. American Philosoph. Society*, Vol. XIII, p. 25–33, N.S. (Philadelphia, 1869). Chase's paper is dated Sept. 18, 1863.

[5] *Op. cit.*, p. 69–72. Oliver's paper is dated May 6, 1864.

[6] *Memoirs American Academy of Arts and Sciences*, Vol. IX, p. 317, N.S. (Cambridge and Boston, 1867).

Of course, I am not prepared to say that these writers never used $n!$ or $\underline{|n}$; I claim only that they usually avoided those symbols. These considerations are a part of the general question of the desirability of the use of symbols in mathematics to the extent advocated by the school of G. Peano in Italy and of A. N. Whitehead and B. Russell in England. The feeling against such a "scab of symbols" seems to be strong and widespread. If the adoption of only one symbol, like our $n!$ or $\underline{|n}$, were involved, the issue would seem trivial, but when dozens of symbols are offered, a more serious situation arises. Certain types of symbols are indispensable; others possess only questionable value. Rich meaning is conveyed instantaneously by $\frac{dy}{dx}$, $\int y dx$, but $\underline{|n}$ and $n!$ serve no other purpose than to save a bit of space. Writers who accept *in toto* the program of expressing all theorems and all reasoning by a severely contracted symbolism must frame notations for matters that can more conveniently be expressed by ordinary words or in less specialized symbolism. We know that intellectual food is sometimes more easily digested, if not taken in the most condensed form. It will be asked, To what extent can specialized notations be adopted with profit? To this question we reply, *only experience can tell.* It is one of the functions of the history of mathematics to record such experiences. Some light, therefore, may be expected from the study of the history of mathematics, as to what constitutes the most profitable and efficient course to pursue in the future. The history of mathematics can reduce to a minimum the amount of future experimentation. Hence algebraic notations deserve more careful historic treatment than they have hitherto received.

450. *Subfactorial "n"* was introduced in 1878 by W. Allen Whitworth[1] and represented by the sign $\underline{||\ \ }$, in imitation of the sign $\underline{|\ \ }$ for factorial n. The value $\underline{||n}$ is obtained by multiplying $\underline{||n-1}$ by n and adding $(-1)^n$. Taking $\underline{||0}=1$, Whitworth gets $\underline{||1}=1-1=0$, $\underline{||2}=0\times2+1=1$, $\underline{||3}=3\times1-1=2$, $\underline{||4}=4\times2+1=9$. Since $\frac{1}{e}=1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+$, etc., Whitworth obtains,[2] when n is even, $\frac{\underline{||n}}{\underline{|n}}>\frac{1}{e}$; when n is odd, $\frac{\underline{||n}}{\underline{|n}}<\frac{1}{e}$; $\lim_{n=\infty}\frac{\underline{||n}}{\underline{|n}}=\frac{1}{e}$. Subfactorials chiefly occur in connection

[1] W. Allen Whitworth, *Messenger of Mathematics*, Vol. VII (1878), p. 145 See also Whitworth, *Choice and Chance* (Cambridge, 1886), Preface, p. xxxiii.

[2] See also *Choice and Chance* (1886), p. 108.

with permutations. Chrystal[1] uses in place of Whitworth's $\underline{n|}$ the sign $n!$

451. *Continued products.*—An infinite product occurs in the writings of Vieta,[2] but without any symbolism to mark that the number of factors is infinite. As reproduced in his collected works of 1646, four factors are written down as follows:

"$\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}$, in $\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}}$, in $\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}}}$, in
$\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{1}{2}}}}}}$."

The factors are separated by a comma and the preposition "in." The extension of the product to one that is infinite is expressed in the words, "in infinitum observata uniformi methodo." Continued products appear in Wallis' *Arithmetica infinitorum* (1655), in passages like the following (p. 179):

"invenietur

$$\square\begin{cases}\text{minor quam } \dfrac{3\times3\times5\times5\times7\times7\times9\times\ 9\times11\times11\times13\times13}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12\times12\times14}\times\sqrt{1\tfrac{1}{13}}\\[2ex] \text{major quam } \dfrac{3\times3\times5\times5\times7\times7\times9\times\ 9\times11\times11\times13\times13}{2\times4\times4\times6\times6\times8\times8\times10\times10\times12\times12\times14}\times\sqrt{1\tfrac{1}{14}}\end{cases}\text{,"}$$

where $\square$ stands for our $\frac{4}{\pi}$. Later (p. 180) Wallis makes the products in the numerator and denominator infinite by adding "&c." as follows: "Dicimus, fractionem illam $\dfrac{3\times3\times5\times5\times7\times7\times\&c.}{2\times4\times4\times6\times6\times8\times\&c.}$ in infinitum continuatam, esse ipsissimum quaesitum numerum $\square$ praecise."

Gauss in 1812 introduced capital Π in the designation of continued products in his paper on the hypergeometric series[3] bearing the title "Disquisitiones generales circa seriem infinitam: $1+\dfrac{\alpha\beta}{1\cdot\gamma}x+\dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx+\dfrac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+$ etc.," which series he represents for brevity by the symbolism $F(\alpha,\beta,\gamma,x)$. The passage in which Π first appears is as follows:

"Introducamus abhinc sequentem notationem:

$$\Pi(k,z)=\frac{1\cdot2\cdot3\ \ldots\ .\ k}{(z+1)(z+2)(z+3)\ \ldots\ .\ (z+k)}\,k^z$$

[1] G. Chrystal, *Algebra*, Part II (1889), p. 25.

[2] F. Vieta, *Opera mathematica* (ed. Fr. à Schooten; Leyden, 1646), p. 400.

[3] C. F. Gauss, *Werke*, Vol. III (Göttingen, 1866), p. 123–61; see p. 144.

ubi k natura sua subintelligitur designare integrum positivum." As previously indicated (§ 449), Gauss later arrives at $\Pi z = 1.2.3. \ldots . z$. He obtains further (p. 148), $\Pi(-\frac{1}{2}) = \sqrt{\pi}$. He represents[1] the function $\frac{d \log \Pi z}{dz}$ by ψz. Jacobi[2] uses Π for continued product. Jordan[3] uses the notation $\Pi_r(c^{\pi p^r} - \overline{K}^\pi)^e$ where the multiplication involves the factors resulting from the different values of r. Chrystal[4] represents the product $(1+u_1)(1+u_2) \ldots (1+u_n)$ by $\overset{n}{\Pi}(1+u_n)$, or "simply by P_n." A. Pringsheim[5] writes $\overset{n}{\underset{0}{\Pi}}(1+u_v)$ to indicate the $(n+1)$ factors from $(1+u_0)$ to $(1+u_n)$.

452. *Permutations and combinations.*—A. T. Vandermonde[6] gives $[n]^p$ as the number of arrangements of n distinct elements, taken p at a time. We have seen (§ 439) that Euler marked the number of combinations of n distinct things taken p at a time by $\left(\frac{n}{p}\right)$ in 1778 and by $\left[\frac{n}{p}\right]$ in 1781, and that nineteenth-century writers changed this to $\binom{n}{p}$. In English textbooks on algebra special symbols for such expressions were slow in appearing. Usually authors preferred to present the actual indicated products and quotients of products. Todhunter[7] uses no special symbolism; Peacock[8] introduces C_r for the combinations of n things taken r at a time, only in deriving the binomial formula and after he had given a whole chapter on permutations and combinations without the use of special notations. However, in 1869 and earlier, Goodwin[9] of Cambridge used ${}_nP_r$ for the number of permutations of n things taken r at a time, so that ${}_nP_n$ was equal to Jarret's $\underline{|n}$ or Kramp's $n!$ Potts[10] begins his treatment by letting the number of

[1] C. F. Gauss, *Werke*, Vol. III, p. 201.

[2] C. G. J. Jacobi, *Gesammelte Werke*, Vol. I (Berlin, 1881), p. 95.

[3] C. Jordan, *Traité des substitutions* (Paris, 1870), p. 430.

[4] G. Chrystal, *Algebra*, Part II (1889), p. 135.

[5] *Encyklopädie d. Math. Wissensch.*, Vol. I, Part I (Leipzig, 1898–1904), p. 113.

[6] *Histoire de l'acad. d. sciences* (Paris, 1772), I, *Mém.*, p. 490.

[7] Isaac Todhunter, *Algebra* (new ed.; London, 1881), p. 286–97.

[8] George Peacock, *Treatise of Algebra* (Cambridge, 1830), chap. ix, p. 200–253. C_r is introduced on p. 255.

[9] Harvey Goodwin, *Elementary Course of Mathematics* (3d ed.; Cambridge, 1869), p. 75, 76

[10] Robert Potts, *Elementary Algebra*, Sec. XI (London, 1880), p. 1–8.

combinations of n different things taken r at a time be denoted by ${}^nC_r$, the number of variations by ${}^nV_r$, the number of permutations by ${}^nP_r$. Whitworth[1] uses P_r^n, C_r^n; also R_r^n for the number of ways of selecting r things out of n when repetitions are allowed; $C_{m,n}$ for the number of orders in which m gains and n losses occur; $J_{m,n}$ for the number of orders in which losses never exceed gains. Chrystal[2] writes ${}_nP_r$, ${}_nC_r$; also ${}_nH_r$ for the number of r combinations of n letters when each letter may be repeated any number of times up to r.

In Germany symbolism appeared in textbooks somewhat earlier. Martin Ohm[3] lets P stand for permutations; $\overset{6}{P}\,(aabccd)$ for the permutations of the six elements a, a, b, c, c, d; V for variations without repetition; V' for variations with repetition; $\overset{\overset{n}{V'}}{(a,\ b,\ c, \ldots .)}$ for the variations of elements $a, b, c, \ldots .$, with repetition, and taken n at a time; $\overset{\overset{n}{C}}{(a,\ b,\ c,\ d, \ldots .)}$ for the combinations of $a, b, c, d, \ldots .$, taken n at a time; C' for combinations with repetition; $Ns.$ for *numerus specierum*; $Ns.\overset{m}{P'}(a^\alpha, b^\beta, c^\gamma, \ldots .)$ for the number of permutations of m elements when a is repeated α times, b is repeated β times, etc.; $Ns.\underset{(a,\ b,\ c\ \ldots .)}{\overset{n}{C}}$ for the number of combinations without repetition, of m elements taken n at a time. A little later (p. 43) Ohm introduces the symbolisms ${}^m\overset{n}{V'}$, ${}^m\overset{n}{C'}$, where m gives the sum of all the n elements appearing in each assemblage. Thus $\overset{{}^7\overset{4}{C'}}{(0,\ 1,\ 2,\ 3, \ldots .)}$ means the combinations, with repetition, which can arise from the elements $0, 1, 2, 3, \ldots .$, when each assemblage contains four elements. There are eleven combinations, viz., 0007, 0016, 0025, 0034, 0115, etc., the sum of the elements in each being 7.

Similar symbolisms are used by Wiegand[4] in 1853 and M. A. Stern[5] in 1860. Thus, Stern represents by ${}^r\overset{m}{C}(1,\ 2 \ldots . n)$ the combinations of the mth class (m elements taken each time), formed from

[1] W. A. Whitworth, *Choice and Chance* (Cambridge, 1886), p. 121.

[2] G. Chrystal, *Algebra*, Part II (Edinburgh, 1899), p. 10.

[3] Martin Ohm, *op. cit.*, Vol. II (Berlin, 1829), p. 33–38.

[4] August Wiegand, *Algebraische Analysis* (2d ed.; Halle, 1853), p. 2–8.

[5] M. A. Stern, *Lehrbuch der algebraischen Analysis* (Leipzig und Heidelberg, 1860), p. 19.

the elements 1, 2, ... , n, such that the sum of the elements in each combination is r; while ${}^{r}\overset{m}{C}(1m, 2m, 3m, \ldots, nm)$ means that each element may be repeated without restriction. E. Netto[1] in 1901 lets $P(a, b, \ldots, g)$, or simply P_n, be the number of permutations of n distinct elements; $P_n[a, b, \ldots, g]$ the assemblage of all the permutations themselves; $I_\kappa^{(n)}$ the number of permutations of n elements which have κ inversions (p. 94). He lets also (p. 119)

$$\begin{array}{ll} C^{(k)}(\int m;\, a_1, a_2, \ldots, a_n) & \Gamma^{(k)}(\int m;\, a_1, a_2, \ldots, a_n) \\ C^{(k)}[\int m;\, a_1, a_2, \ldots, a_n] & \Gamma^{(k)}[\int m;\, a_1, a_2, \ldots, a_n] \\ V^{(k)}(\int m;\, a_1, a_2, \ldots, a_n) & \Phi^{(k)}(\int m;\, a_1, a_2, \ldots, a_n) \\ V^{(k)}[\int m;\, a_1, a_2, \ldots, a_n] & \Phi^{(k)}[\int m;\, a_1, a_2, \ldots, a_n] \end{array}$$

stand for the combinations and variations of the numbers $a_1, a_2, \ldots, a_n$, taken (k) at a time, and yielding in each assemblage the prescribed sum m; where the four symbols on the left allow no repetition, while the four symbols on the right allow unrestricted repetition, and where the round parentheses indicate the number of assemblages, and the brackets the collection of all assemblages themselves connected by the $+$ sign.

In the *Encyclopédie* (1904)[2] the number of arrangements (permutations) of n things taken p at a time is marked by A_n^p, the number of combinations by C_n^p and by $\binom{n}{p}$. Moreover,

$$\Gamma_n^p \equiv n(n+1) \ldots (n+p-1) \div p!\,.$$

453. *Substitutions.*—Ruffini,[3] in his theory of equations of 1799, introduces a symbolism for functions designed for the consideration of their behavior when subjected to certain substitutions. He writes:

$f(x)$ for any function of x.

$f(x)(y)$ for any function of x and y.

$f(x, y)$ for a function which retains its value when x and y are permuted, as $x^3+y^3+z^3=f(x, y, z)$.

$f(x, y)(z)$ for a function like x^2+y^2+z.

$f((x)(y), (z)(u))$ for a function like $\frac{x}{y}+\frac{z}{u}$ which remains the same in value when x and z interchange simultaneously when y and u interchange.

[1] Eugen Netto, *Lehrbuch der Combinatorik* (Leipzig, 1901), p. 2, 3.

[2] *Encyclopédie d. scien. math.*, Tom. I, Vol. I (1904), p. 67, article by E. Netto, revised by H. Vogt.

[3] Paolo Ruffini, *Teoria generale delle equazioni* (Bologna, 1799), p. 1–6.

$f((x, y), (z, u))$ for a function like $xy+zu$, which remains invariant when x and y, or z and u, are interchanged, or when both x and y are changed into z and u.

If $Y=f(r)(s)(t)(u)\ldots.(z)$, say $=r^3x^3-4r^2st+rstu-2s^2tx+4ux^2$,

then, $r|Y$ represents the aggregate of terms in Y which contain r,

$r\left|\frac{s}{Y}\right.$ represents the aggregate of all the terms which contain r and do not contain s.

Cauchy[1] in 1815 represented a substitution in this manner:

$$\begin{pmatrix}1 & 2 & 3 & . & . & . & . & n\\ \alpha & \beta & \gamma & . & . & . & . & \zeta\end{pmatrix}$$

where every number of the upper line is to be replaced by the letter of the lower line immediately below.

Jordan[2] represents a substitution, S, by the notation $S=(ab\ldots.k)(a'b'\ldots.k')(a''\ldots.k'')\ldots.$ where each parenthesis exhibits a cycle of substitutions. If A and B are two substitutions, then Jordan lets AB represent the result of the substitution A, followed by the substitution B, and the meanings of $A^2, A^3, \ldots., A^{-1}$ are evident.

E. Netto[3] writes substitutions in the form similar to that of Cauchy,

$$\begin{pmatrix}x_1, & x_2, & x_3, & \ldots., & x_n\\ x_{i_1}, & x_{i_2}, & x_{i_3}, & \ldots., & x_{i_n}\end{pmatrix}.$$

Netto uses also the form of cycles employed by Jordan. If A and B are two substitutions, and B^{-1} is the inverse of A, then the substitution $B^{-1}AB$ signifies the transformation[4] of A by B. Some writers—for instance, W. Burnside[5]—indicate the inverse substitution of s by s_{-1}. If s, t, v are substitutions,[6] then in their product stv the order of

[1] A. L. Cauchy, *Journal de l'école polyt.*, Tome X, Cah. 17, p. 56. Reference taken from Thomas Muir, *The Theory of Determinants in the Historical Order of Development*, Vol. I (London; 2d ed., 1906), p. 101.

[2] Camille Jordan, *Traité des substitutions* (Paris, 1870), p. 21.

[3] Eugen Netto, *Substitutionstheorie* (Leipzig, 1882); English translation by F. N. Cole (Ann Arbor, 1892), p. 19.

[4] C. Jordan, *op. cit.*, p. 23.

[5] W. Burnside, *Theory of Groups of Finite Order* (Cambridge, 1897), p. 4.

[6] L. E. Dickson, *Theory of Algebraic Equations* (New York, 1903), p. 11. See also Harold Hilton, *Introduction to the Theory of Groups of Finite Order* (Oxford, 1908), p. 12, 13.

applying the factors was from right to left with Cayley and Serret, but in the more modern use the order is from left to right, as in Jordan's work.

By the symbolism $\left[\begin{matrix} f(z) \\ z \end{matrix}\right]$ or simply $f(z)$, there is represented in J. A. Serret's *Algebra*[1] the substitution resulting from the displacing of each index z by the function $f(z)$, where z is given in order n values, which it is assumed agree, except in order, with the simultaneous values of $f(z)$. Similarly,[2] $\left(\begin{matrix} az+b \\ z \end{matrix}\right)$ represents the $n(n-1)$ linear substitutions, where z takes in succession all the indices $0, 1, 2, \ldots,$ $(n-1)$ of the n variables $x_0, x_1, \ldots, x_{n-1}$, and the values of $az+b$ are taken with respect to modulus of the prime n between the limits, 0 and $n-1$.

454. *Groups.*—The word "group" in a technical sense was first used by E. Galois in 1830. He[3] first introduced the mark ∞ into group theory by the statement that in the group $x_{\frac{k}{l}}, x_{\frac{ak+bl}{ck+dl}}$, "$\frac{k}{l}$ peut avoir les $p+1$ valeurs $\infty, 0, 1, 2, \ldots, p-1$. Ainsi, en convenant que k peut être infini, on peut écrire simplement $x_k, x_{\frac{ak+b}{ck+d}}$." In another place[4] Galois speaks of substitutions "de la forme $(a_{k_1, k_2}, a_{mk_1+nk_2})$." Again he represents a substitution group by writing down the substitutions in it,[5] as "le groupe *abcd, badc, cdab, dcba.*"

Jordan[6] represents the group formed by substitutions $A, B, C,$ $\ldots,$ by the symbol $(A, B, C, \ldots)$, but frequently finds it convenient to represent a group by a single letter, such as F, G, H, or L.

Certain groups which have received special attention have been represented by special symbols. Thus, in groups of movement, the group formed by rotations about a single line is the cyclic group[7] C_a, the group formed by rotations through π about three mutually perpendicular intersecting lines is the quadratic group D, rotations still differently defined yield the tetrahedral group T, the octahedral group O, the

[1] J. A. Serret, *Handbuch der höheren Algebra* (trans. by G. Wertheim), Vol. II (2d ed.; Leipzig, 1879), p. 321.

[2] J. A. Serret, *op. cit.*, Vol. II, p. 353.

[3] E. Galois, *Œuvres* (ed. E. Picard; Paris, 1897), p. 28.

[4] *Op. cit.*, p. 54.

[5] *Op. cit.*, p. 45.

[6] C. Jordan, *op. cit.*, p. 22, 395.

[7] See Harold Hilton, *op. cit.* (Oxford, 1908), p. 113–15.

icosahedral group E. The consideration of rotary inversions leads to cyclic groups c_m, generated by an m-al rotary inversion about a point O and a line l, and to groups marked D_m, d_m, and so on. Hilton's symbolism is further indicated in such phrases[1] as "suppose that $\{a\}$ is a normal cyclic subgroup of the Group $G \equiv \{a, b\}$ generated by the elements a and b."

Let S_1, S_2, S_3, ... , be a given set of operations, and G_1, G_2, ... , a set of groups, then the symbol $\{S_1, S_2, S_3, \ldots\ldots, G_1, G_2, \ldots\ldots\}$ is used by W. Burnside to denote the group that arises by combining in every possible way the given operations and the operations of the given groups.[2]

The notation $\frac{G}{G'}$ was introduced by Jordan[3] in 1873 and then used by Hölder[4] and others to represent a "factor-group" of G, or the "quotient" of G by G'.

The symbol $\vartheta(P)$ has been introduced by Frobenius[5] to denote $(p-1)(p^2-1)\ \ldots\ldots\ (p^{\kappa}-1)$, where κ is the greatest of a series of exponents of p.

455. In the theory of continuous groups single letters T or U are used to designate transformations, T^{-1}, U^{-1} being inverse transformations. The notation TU was employed by Sophus Lie[6] to represent the product of T and U, the transformation T being followed by the transformation U. If (x) and (x') are points such that $(x')=T(x)$, then $(x)=T^{-1}(x')$. In the relation $(y')=TUT^{-1}(x')$ one has the symbolic notation for "the transformation of U by T." The index of a subgroup H of a group G is often represented by the symbol (G, H). G. Frobenius[7] uses this same symbol in a more general sense.

The need of symbolism in the study of groups made itself felt in applications to crystallography. A. Schoenflies[8] marks a revolution about an axis a through an angle α by $\mathfrak{A}(\alpha)$ or simply by $\mathfrak{A}$. For

[1] *Op. cit.*, p. 169.

[2] W. Burnside, *Theory of Groups of Finite Order* (Cambridge, 1897), p. 27.

[3] C. Jordan, *Bulletin Soc. math.* (1st ser.), Vol. I (1873), p. 46.

[4] O. Hölder, *Mathematische Annalen*, Vol. XXXIV (1889), p. 31.

[5] G. Frobenius, *Sitzungsberichte d. k. p. Akademie der Wissensch.* (Berlin, 1895), p. 1028.

[6] S. Lie and F. Engel, *Theorie der Transformationsgruppen*, Vol. I (Leipzig, 1888), p. 223. See also *Encyclopédie des scien. math.*, Tom. II, Vol. IV, p. 162–63.

[7] G. Frobenius, *Crelle's Journal*, Vol. CI (1887), p. 273.

[8] Arthur Schoenflies, *Krystallsysteme und Krystallstructur* (Leipzig, 1891), p. 32, 36.

brevity, $\mathfrak{A}(n\alpha)$ is written $\mathfrak{A}^n$. A reflection (*Spiegelung*) is marked by $\mathfrak{S}$, an operation of the second kind (involving a reflection and a rotation) having the axis a and the angle α is marked $\overline{\mathfrak{A}}(\alpha)$ or $\overline{\mathfrak{A}}$. The cyclic group of rotation is marked C_n, n indicating the number of rotations;[1] the Dieder (regular n gon) group by D_n, which is equal to $\left\{ \mathfrak{A}\left(\frac{2\pi}{n}\right), \mathfrak{U} \right\}$ where $\mathfrak{U}$ indicates an inversion (*Umklappung*). In common with other writers, Schoenflies indicates the tetrahedral group by T and the octahedral by O. In the treatment of these groups and the corresponding classes of crystals, he[2] uses h_p or l_p as a p-fold axis of symmetry, thereby following the notation of Bravais,[3] except that the latter uses capital letters for the designation of lines. Schoenflies[4] marks translation groups by Γ_τ and space group by Γ. On pages 555, 556, he gives a table of space groups (32 in all), with their individual symbols, for the various classes of crystals.

456. *Invariants and covariants.*—In the treatment of this subject Cayley[5] let $\xi_1=\delta_{x_1}$, $\eta_1=\delta_{y_1}$, where δ_{x_1} and δ_{y_1} are symbols of differentiation relative to x_1 and y_1, and similarly for the more general x_p, y_p, and he writes $\xi_1\eta_2-\xi_1\eta_1=\overline{12}$, etc.; the result of operating r times with $\overline{12}$ he marks $\overline{12}^r$. For the Caleyan notation Aronhold, Clebsch,[6] and Gordan[7] advanced one of their own based on a symbolical representation of the coefficients in a quantic. A binary quantic $a_0x^n+na_1x^{n-1}+\ldots\ldots+a_n$ is represented by $(a_1x+a_2y)^n$. The quantic of the nth degree in three variables x_1, x_2, x_3 is symbolically written $(a_1x_1+a_2x_2+a_3x_3)^n$ or a_x^n, where a_1, a_2, a_3 are umbral symbols not regarded as having any meaning separately.

457. *Dual arithmetic.*—Perhaps the following quotations will convey an idea of the sign language introduced by Byrne in his *Dual*

[1] *Op. cit.*, p. 58, 61. [2] *Op. cit.*, p. 72, 73.

[3] A. Bravais, "Mémoire sur les polyèdres de forme symétrique," *Journal de math. par Liouville*, Vol. XIV (Paris, 1849), p. 141–80. We may add here that Bravais' "symbole de la symétrie du polyèdre" (24 symbols in all) are tabulated on p. 179 of his article.

[4] *Op. cit.*, p. 301, 359, 362.

[5] A. Cayley, *Cambridge and Dublin Mathematical Journal*, Vol. I (1846), p. 104–22; *Collected Mathematical Papers*, Vol. I, p. 96, 97. See also § 462.

[6] R. F. A. Clebsch, *Theorie der Binären Formen.*

[7] P. Gordan, *Vorlesungen über Invariantentheorie.* See also G. Salmon, *Modern Higher Algebra* (3d ed.; Dublin, 1876), p. 266, 267; Faà de Bruno, *Théorie des formes binaires* (Turin, 1876), p. 293; E. B. Elliott, *Algebra of Quantics* (Oxford, 1913), p. 68.

Arithmetic. He says: ". . . . the continued product of $1{\cdot}2345678 \times (1{\cdot}01)^4(1{\cdot}001)^5(1{\cdot}0001)^6$ which may be written $1{\cdot}2345678 \downarrow 0, 4, 5, 6, = 1{\cdot}291907$. The arrow $\downarrow$ divides the coefficient $1{\cdot}2345678$ and the powers of $1{\cdot}1$, $1{\cdot}01$, $1{\cdot}001$, $1{\cdot}0001$, &c; 0, immediately follows the arrow because no power of $1{\cdot}1$ is employed; 6, is in the fourth place after $\downarrow$, and shows that this power operated upon periods of four figures each; 5 being in the third place after $\downarrow$ shows by its position, that its influence is over periods of three figures each; and 4 occupies the second place after $\downarrow$ . . . . ."[1] A few years later Byrne[2] wrote: "Since any number may be represented in the form $N = 2^n 10^m \downarrow u_1, u_2, u_3, u_4, u_5$, etc., we may omit the bases 2 and 10 with as much advantage in perspicuity as we omitted the bases $1{\cdot}1$, $1{\cdot}01$, $1{\cdot}001$, &c. and write the above expression in the form $N = {}^m\!\downarrow{}^n u_1, u_2, u_3, u_4$, etc." "Any number N may be written as a continued product of the form $10^m \times (1-{\cdot}1)^{v_1}(1-{\cdot}01)^{v_2}(1-.001)^{v_3}$ $(1-.0001)^{v_4}$, etc." or $10^m({\cdot}9)^{v_1}({\cdot}99)^{v_2}({\cdot}999)^{v_3}({\cdot}9999)^{v_4}$, etc." "In analogy with the notation used in the descending branch of dual arithmetic, this continued product may be written thus $'v_1'v_2'v_3'v_4'v_5'v_6\downarrow^m$ where any of the digits v_1, v_2, v_3, etc. as well as m may be positive or negative."[3]

"As in the ascending branch, the power of 10,m, may be taken off the arrow and digits placed to the right when m is a $+$ whole number. Thus $'v_1\ 'v_2\ 'v_3\ 'v_4 \uparrow t_1'\ t_2'\ t_3'$ etc. represents the continued product $({\cdot}9)^{v_1}$ $({\cdot}99)^{v_2}({\cdot}999)^{v_3}({\cdot}9999)^{v_4}(9)^{t_1}(99)^{t_2}(999)^{t_3}$." Dual signs of addition $\ddagger$ and subtraction $\rightarrow$ are introduced for ascending branches; the reversion of the arrows gives the symbols for descending branches.

458. *Chessboard problem.*—In the study of different non-linear arrangements of eight men on a chessboard, T. B. Sprague[4] lets each numeral in 61528374 indicate a row, and the position of each numeral (counted from left to right) indicates a column. He lets also i denote "inversion" so that $i(61528374) = 47382516$, r "reversion" (each number subtracted from 9) so that $r(61528374) = 38471625$, p "perversion" (interchanging columns and rows so that the 6 in first column becomes 1 in the sixth column, the 1 in the second column becomes 2 in the first column, the 5 in the third column becomes 3 in the fifth column, etc., and then arranging in the order of the new columns) so that $p(61528374) = 24683175$. It is found that $i^2 = r^2 = p^2 = 1$, $ir = ri$, $ip = pr$, $rp = pi$, $irp = rip = ipi$, $rpr = pir = pri$.

[1] Oliver Byrne, *Dual Arithmetic* (London, 1863), p. 9.

[2] Oliver Byrne, *op. cit.*, Part II (London, 1867), p. v.

[3] O. Byrne, *op. cit.*, Part II (1867), p. x.

[4] T. B. Sprague, *Proceedings Edinburgh Math. Soc.*, Vol. VIII (1890), p. 32.

DETERMINANT NOTATIONS

459. *Seventeenth century.*—The earliest determinant notations go back to the originator of determinants in Europe, Leibniz,[1] who, in a letter of April 28, 1693, to Marquis de l'Hospital, writes down the three equations

$$\begin{aligned} 10+11x+12y&=0\,, \\ 20+21x+22y&=0\,, \\ 30+31x+32y&=0\,. \end{aligned}$$

In this topographic notation the coefficients 10, 11, 12, etc., are to be interpreted as indices like the subscripts in our modern coefficient notation a_{10}, a_{11}, a_{12}, etc. Eliminating y and then x, Leibniz arrives at the relation

$$\begin{matrix} 1_0\ 2_1\ 3_2 & & 1_0\ 2_2\ 3_1 \\ 1_1\ 2_2\ 3_0 & \textit{aeqv.} & 1_1\ 2_0\ 3_2 \\ 1_2\ 2_0\ 3_1 & & 1_2\ 2_1\ 3_0\,, \end{matrix}$$

where $1_0\ 2_1\ 3_2$ represents a product of the three coefficients and, generally, each of the six rows represents such a product (§ 547).

460. *Eighteenth century.*—Independently of Leibniz, determinants were reinvented by Gabriel Cramer[2] who writes the coefficients of linear equations Z^1, Y^2, X^3, ... the letters indicating the column, the superscripts the rows, in which the coefficients occur. Cramer says: "Let there be several unknowns $z, y, x, v, \ldots.$, and also the equations

$$\begin{aligned} A^1&=Z^1z+Y^1y+X^1x+V^1v+\ldots.\,, \\ A^2&=Z^2z+Y^2y+X^2x+V^2v+\ldots.\,, \\ A^3&=Z^3z+Y^3y+X^3x+V^3v+\ldots.\,, \\ A^4&=Z^4z+Y^4y+X^4x+V^4v+\ldots.\,, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where the letters A^1, A^2, A^3, A^4, ... , do not as usual denote the powers of A, but the left side of the first, second, third, fourth, ... , equation, assumed to be known. Likewise Z^1, Z^2, ... , are the coefficients of z, Y^1, Y^2, ... , those of y, X^1, X^2, ... , those of x, V^1, V^2, ... , those of v, ... , in the first, second, ... , equation. This no-

[1] *Leibnizens Mathematische Schriften* (ed. C. I. Gerhardt, 3F.), Vol. II (1850), p. 229, 234, 239, 240, 241, 245, 261; Vol. V, p. 348. See also Vol. I, p. 120, 161; Vol. II, p. 7, 8; *Bibliotheca mathematica* (3d ser.), Vol. XIII (1912–13), p. 255; S. Günther, *Lehrbuch der Determinantentheorie* (Erlangen, 1877), p. 2, 3.

[2] Gabriel Cramer, *Introduction à l'analyse des lignes courbes algébriques* (Geneva, 1750), p. 657–59.

tation being adopted, one has, if only one equation with one unknown z is presented, $z=\frac{A^1}{Z^1}$. When there are two equations and two unknowns x and y, one finds

$$z=\frac{A^1Y^2-A^2Y^1}{Z^1Y^2-Z^2Y^1}\,,$$

and

$$y=\frac{Z^1A^2-Z^2A^1}{Z^1Y^2-Z^2Y^1}\,."$$

His method of solving simultaneous linear equations attracted the attention of Bézout,[1] Vandermonde,[2] and Laplace. Bézout lets (ab') stand for $ab'-a'b$; he similarly denotes a determinant of the third order $(ab'c'')$. An example of Vandermonde's symbols is seen in the two equations

$$\begin{aligned} &{}^1_1\xi_1+{}^1_2\xi_2+{}^1_3=0\,,\\ &{}^2_1\xi\;+{}^2_2\xi_2+{}^2_3=0\,, \end{aligned}$$

in which the designation of coefficients by numerical superscripts and subscripts resembles the device of Leibniz. Furthermore, Vandermonde in 1772 adopts the novel notation

$$\frac{\alpha\,|\,\beta}{a\,|\,b}\text{ for }a^\alpha\cdot b^\beta-b^\alpha\cdot a^\beta\,,$$

and

$$\frac{\alpha\,|\,\beta\,|\,\gamma}{a\,|\,b\,|\,c}\text{ for }a^\alpha\cdot\frac{\beta\,|\,\gamma}{b\,|\,c}+b^\alpha\cdot\frac{\beta\,|\,\gamma}{c\,|\,a}+c^\alpha\cdot\frac{\beta\,|\,\gamma}{a\,|\,b}\,,$$

and finds

$$\frac{\alpha\,|\,\beta}{a\,|\,b}=-\frac{\alpha\,|\,\beta}{b\,|\,a}\,.$$

"Le symbole $\dashv\!\vdash$ sert ici de charactéristique."

Laplace[3] expressed the resultant of three equations by $({}^1a\cdot{}^2b\cdot{}^3c)$, those indices being placed to the left of a letter and above.

461. *Early nineteenth century.*—The first use of the term "determinant" occurs in the following passage of K. F. Gauss: "Nume-

[1] E. Bézout, *Théorie générale des équations algébriques* (Paris, 1779), p. 211, 214.

[2] A. T. Vandermonde, *Histoire de l'acad. des scienc. Paris*, année 1772 (Paris, 1775), p. 516.

[3] P. S. Laplace, *Histoire de l'acad. des scienc. Paris*, année 1772 (2d partie), p. 267, 294; *Œuvres*, Vol. VIII, p. 365–406.

rum $bb-ac$, a cuius indole proprietates formae (a, b, c) imprimis pendere in sequentibus docebimus, *determinantem* huius formae uocabimus."[1]

The notation of Binet[2] is seen in the following quotation from his discussion of the multiplication theorem: "Désignons par $S(y', z'')$ une somme de résultantes, telle que $(y'_{,}, z''_{,})+(y'_{,,}, z''_{,,})+(y'_{,,,}, z''_{,,,})+$ etc. c'est-à-dire, $y'_{,}z''_{,}-z'_{,}y''_{,}+y'_{,,}z''_{,,}-z'_{,,}y''_{,,}+y'_{,,,}z''_{,,,}-z'_{,,,}y''_{,,,}+$etc.; et continuons d'employer la charactéristique Σ pour les intégrales relatives aux accens supérieurs des lettres. L'expression $\Sigma[S(y,z')\cdot S(v, \zeta')]$ devient. ..."

Cauchy[3] represents alternating functions by the symbolism $S(\pm a_1^1 a_2^2 a_3^3 \ldots . a_n^n)$ or, writing $a_{r\cdot s}$ for a_r^s, the symbolism

$$S(\pm a_{1\cdot 1} a_{2\cdot 2} a_{3\cdot 3} \ldots . a_{n\cdot n}) .$$

Cauchy represents by $(a_{1\cdot P}^{(p)})$ "un système symmétrique de l'ordre P," embracing $P\times P$ determinants which may be arranged in a square as follows:[4]

$$\left\{\begin{array}{l} a_{1\cdot 1}^{(p)}\, a_{1\cdot 2}^{(p)} \ldots . a_{1\cdot P}^{(p)} \\ a_{2\cdot 1}^{(p)}\, a_{2\cdot 2}^{(p)} \ldots . a_{2\cdot P}^{(p)} \\ \ldots\ldots\ldots\ldots \\ a_{P\cdot 1}^{(p)}\, a_{P\cdot 2}^{(p)} \ldots . a_{P\cdot P}^{(p)} , \end{array}\right.$$

where $P=n(n-1) \ldots . (n-p+1)/1\cdot 2\cdot 3 \ldots . p$. Of these determinants, $a_{1\cdot 1}^{(p)}$, for example, would be in our modern notation:

$$\left|\begin{array}{l} a_{1\cdot 1}\, a_{1\cdot 2} \ldots . a_{1\cdot p} \\ a_{2\cdot 1}\, a_{2\cdot 2} \ldots . a_{2\cdot p} \\ \ldots\ldots\ldots\ldots \\ a_{p\cdot 1}\, a_{p\cdot 2} \ldots . a_{p\cdot p} \end{array}\right| .$$

[1] C. F. Gauss, *Disquisitiones arithmeticae* (1801), XV, p. 2; Thomas Muir, *The Theory of Determinants in the Historical Order of Development* (London; 2d ed., 1906), p. 64.

[2] Jacques P. M. Binet, *Jour. de l'école polyt.*, Tome IX, Cahier 16 (1813), p. 288; T. Muir, *op. cit.*, Vol. I (1906), p. 84.

[3] A. L. Cauchy, *Jour. de l'école polyt.*, Tome X, Cahier 17 (1815), p. 52; *Œuvres*, IIe Série, Vol. I, p. 113, 114, 155, 156.

[4] Th. Muir, *op. cit.*, Vol. I (1906), p. 111, 112.

Desnanot[1] in 1819, instead of writing indices in the position of exponents, places them above the letters affected, as in $\overset{k}{a}$, where k is the index of a. He writes for $\overset{k}{a}\overset{l}{b}-\overset{k}{b}\overset{l}{a}$ the abbreviation $(\overset{k}{a}\overset{l}{b})$.

By the notation $P\left(\underset{n}{\overset{n}{a}};\, \underset{h}{s},\, \underset{h}{\overset{1}{a}}\right)$ Scherk[2] represents

$$\left|\begin{array}{ccccc} \underset{1}{s} & \underset{1}{\overset{2}{a}} & \underset{1}{\overset{3}{a}} & \ldots\ldots & \underset{1}{\overset{n}{a}} \\ \underset{2}{s} & \underset{2}{\overset{2}{a}} & \underset{2}{\overset{3}{a}} & \ldots\ldots & \underset{2}{\overset{n}{a}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \underset{n}{s} & \underset{n}{\overset{2}{a}} & \underset{n}{\overset{3}{a}} & \ldots\ldots & \underset{n}{\overset{n}{a}} \end{array}\right|$$

The notation of Schweins[3] is a modification of that of Laplace. Schweins uses $\|\ )$ where Laplace uses $(\ )$. Schweins writes $\left\|\overset{a_1}{A_1}\right)=\overset{a_1}{A_1}$, and in general

$$\left\|\begin{matrix} a_1 \ldots\ldots a_n \\ A_1 \ldots\ldots A_n \end{matrix}\right) = \sum (-1)^x \left\|\begin{matrix} a_1 \ldots\ldots a_{n-x-1}\, a_{n-x+1} \ldots\ldots a_n \\ A_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A_{n-1} \end{matrix}\right) \cdot \overset{a_{n-x}}{A_{n}}, \qquad x=0,\, 1,\, \ldots\ldots,\, n-1\,.$$

Reiss[4] adopts in 1829 the notation $(abc, \overline{123})$ for $a^1b^2c^3-a^1b^3c^2-a^2b^1c^3+a^2b^3c^1+a^3b^1c^2-a^3b^2c^1$, the line above the 123 is to indicate permutation. He uses the same notation[5] in 1838.

[1] P. Desnanot, *Complément de la théorie des équations du premier degré* (Paris, 1819); Th. Muir, *op. cit.*, Vol. I (1906), p. 138.

[2] H. F. Scherk, *Mathematische Abhandlungen* (Berlin, 1825), 2. Abhandlung; Th. Muir, *op. cit.*, Vol. I (1906), p. 151, 152.

[3] Ferd. Schweins, *Theorie der Differenzen und Differentiale* (Heidelberg, 1825); *Theorie der Producte mit Versetzungen*, p. 317–431; Th. Muir, *op. cit.*, Vol. I (1906), p. 160.

[4] M. Reiss in *Correspondance math. et phys.*, Vol V (1829), p. 201–15; Th. Muir, *op. cit.*, Vol. I (1906), p. 179.

[5] M. Reiss, *Correspondance math. et phys.*, Vol. X (1838), p. 229–90; Th. Muir, *op. cit.*, Vol. I (1906), p. 221.

Jacobi[1] chose the representation $\Sigma \pm a_{r_0, s_0} a_{r_1, s_1} \ldots . a_{r_m, s_m}$ and in 1835 introduced also another:

$$a \begin{Bmatrix} r_0, r_1, r_2, \ldots . r_m \\ s_0, s_1, s_2, \ldots . s_m \end{Bmatrix} .$$

Sometimes he used[2] $(a_{0,0}\, a_{1,1} \ldots . a_{m-1, m-1}\, a_{k,i})$, a notation resembling that of Bézout.

V. A. Lebesgue,[3] in his notation for minors of a determinant D, lets $[g, i]$ represent the determinant left over after the suppression of the row g and the column i, and he lets $\begin{bmatrix} g, i \\ h\ k \end{bmatrix}$ represent the determinant resulting from the suppression of the rows g and h, and the columns i and k.

In 1839, J. J. Sylvester[4] denotes by the name "zeta-ic multiplication," an operation yielding such products as the following:

$$\zeta(a_1 - b_1)(a_1 - c_1) = a_2 - a_1 b_1 - a_1 c_1 + b_1 c_1 .$$

For difference products he writes:

$$(b-a)(c-a)(c-b) = PD(a\ b\ c)$$
$$abc(b-a)(c-a)(c-b) = PD(0\ a\ b\ c) .$$

Combining the two notations, he represents the determinant of the system

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}$$

by $\zeta abcPD(abc)$ or $\zeta PD(0abc)$, and calls it "zeta-ic product of differences." Muir states (Vol. I, p. 230): "Now Sylvester's ζPD notation being unequal to the representation of the determinant $|a_1 b_2 c_4 d_5|$ in which the index-numbers do not proceed by common difference 1, he would seem to have been compelled to give a periodic character to the arguments of the bases in order to remove the difficulty. At any rate, the difficulty *is* removed; for the number of terms

[1] C. G. J. Jacobi in *Crelle's Journal*, Vol. XV (1836), p. 115; *Werke*, Vol. III, p. 295–320; Th. Muir, *op. cit.*, Vol. I (1906), p. 214.

[2] C. G. J. Jacobi, *Werke*, Vol. III, p. 588, 589.

[3] V. A. Lebesgue in *Liouville's Journal de Math.*, Vol. II (1837), p. 344; Th. Muir, *op. cit.*, Vol. I (1906), p. 220.

[4] J. J. Sylvester in *Philosophical Magazine*, Vol. XVI (1840), p. 37–43; *Collected Math. Papers*, Vol. I, p. 47–53; Th. Muir, *op. cit.*, Vol. I (1906), p. 228, 230.

in the period being 5 the index-numbers 4 and 5 become changeable into -1 and 0 and . . . . . Sylvester's form of the result (in zeta-ic products) thus is $\zeta\{S_2(abcd)\cdot\zeta PD(0abcd)\}=\zeta_{-2}(0abcd)$."

In 1840 Cauchy[1] obtained as a result of elimination "la fonction alternée formée avec les quantités que présente le tableau

$$\left\{\begin{matrix} a, & 0, & A, & 0, & 0, \\ b, & a, & B, & A, & 0, \\ c, & b, & C, & B, & A, \\ d, & c, & 0, & C, & B, \\ 0, & d, & 0, & 0, & C. \end{matrix}\right.$$

462. *Modern notations.*—A notation which has rightly enjoyed great popularity because of its objective presentation of the elements composing a determinant, in convenient arrangement for study, was given in 1841 by Cayley. He says: "Let the symbols

$$|\,\alpha\,|, \quad \begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix}, \quad \begin{vmatrix} \alpha\ , & \beta\ , & \gamma \\ \alpha'\ , & \beta'\ , & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}, \text{ etc.,}$$

denote the quantities

$$\alpha,\ \alpha\beta'-\alpha'\beta,\ \alpha\beta'\gamma''-\alpha\beta''\gamma'+\alpha'\beta''\gamma-\alpha'\beta\gamma''+\alpha''\beta\gamma'-\alpha''\beta'\gamma\ , \text{ etc.,}$$

the law of whose formation is tolerably well known. . . . ."[2] "Here, then," says Muir, "we have for the first time in the notation of determinants, the pair of upright lines so familiar in all the later work. The introduction of them marks an epoch in the history, so important to the mathematician is this apparently trivial matter of notation. By means of them every determinant became representable, no matter how heterogeneous or complicated its elements might be; and the most disguised member of the family could be exhibited in its true lineaments. While the common characteristic of previous notations is their ability to represent the determinant of such a system as

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \quad \text{or} \quad \begin{matrix} a_{1\cdot 1} & a_{1\cdot 2} & a_{1\cdot 3} \\ a_{2\cdot 1} & a_{2\cdot 2} & a_{2\cdot 3} \\ a_{3\cdot 1} & a_{3\cdot 2} & a_{3\cdot 3} \end{matrix}$$

[1] A. L. Cauchy, *Exercices d'analyse et de phys. math.*, Vol. I, p. 385–422; *Œuvres complètes* (2d ser.), Vol. XI, p. 471; Th. Muir, *op. cit.*, Vol. I (1906), p. 242.

[2] A. Cayley, *Cambridge Math. Journal*, Vol. II (1841), p. 267–71; *Collected Math. Papers*, Vol. I, p. 1–4; Th. Muir, *op. cit.*, Vol. II (1911), p. 5, 6.

and failure to represent in the case of systems like

$$\begin{array}{ccc} a\,b\,c & a\,b\,c & 4\,5\,6 \\ c\,a\,b & 1\,a\,b & 3\,2\,7 \\ b\,c\,a, & 0\,1\,a, & 8\,1\,0: \end{array}$$

Cayley's notation is equally suitable for all."

The first occurrence of Cayley's vertical-line notation for determinants and double vertical-line notation for matrices in *Crelle's Journal*[1] is in his "Mémoire sur les hyperdéterminants"; in *Liouville's Journal*, there appeared in 1845 articles by Cayley in which [ ] and { } are used in place of the vertical lines.[2] The notation { } was adopted by O. Terquem[3] in 1848, and by F. Joachimsthal[4] in 1849, who prefixes "dét," thus: "dét. { }." E. Catalan[5] wrote "dét. (A, B, C . . . .)," where A, B, C, ... , are the terms along the principal diagonal. The only objection to Cayley's notation is its lack of compactness. For that reason, compressed forms are used frequently when objective presentation of the elements is not essential. In 1843 Cayley said: "Representing the determinants

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ etc.,}$$

by the abbreviated notation $\overline{123}$, etc.; the following equation is identically true:

$$\overline{345}\cdot\overline{126}-\overline{346}\cdot\overline{125}+\overline{356}\cdot\overline{124}-\overline{456}\cdot\overline{123}=0\ ."^{6}$$

A little later Cayley wrote: "Consider the series of terms

$$\begin{array}{llll} x_1 & x_2 & \ldots\ldots & x_n \\ A_1 & A_2 & \ldots\ldots & A_n \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ K_1 & K_2 & \ldots\ldots & K_n \end{array}$$

[1] A. Cayley, *Crelle's Journal*, Vol. XXX (1846), p. 2.

[2] A. Cayley, *Liouville's Journal*, Vol. X (1845), p. 105 gives [ ], p. 383 gives { }.

[3] *Nouvelles Annales de Math.*, Vol. VII (1848), p. 420; Th. Muir, *op. cit.*, Vol. II, p. 36.

[4] F. Joachimsthal, *Crelle's Journal*, Vol. XL (1849), p. 21–47.

[5] E. Catalan, *Bulletin de l'acad. roy. de Belgique*, Vol. XIII (1846), p. 534–55; Th. Muir, *op. cit.*, Vol. II, p. 37.

[6] A. Cayley, *Cambridge Math. Journal*, Vol. IV (1845), p. 18–20; *Collected Math. Papers*, Vol. I, p. 43–45; Th. Muir, *op. cit.*, Vol. II, p. 10.

the number of quantities $A, \ldots, K$ being equal to q $(q<n)$. Suppose $q+1$ vertical rows selected, and the quantities contained in them formed into a determinant, this may be done in

$$\frac{n(n-1)\ldots\ldots(q+2)}{1\cdot 2\ldots\ldots(n-q-1)}$$

different ways. The system of determinants so obtained will be represented by the notation

$$\left\Vert \begin{matrix} x_1 & x_2 \ldots\ldots & x_n \\ A_1 & A_2 \ldots\ldots & A_n \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ K_1 & K_2 \ldots\ldots & K_n \end{matrix} \right\Vert ,$$ [1]

If this form is equated to zero, it signifies that each of these determinants shall be equated to zero, thus yielding a system of equations.

A function U, from which he considers three determinants to be derived,[2] is

$$\begin{array}{l} x(\alpha\xi+\beta\eta+\ldots\ldots) \\ +y(\alpha'\xi+\beta'\eta+\ldots\ldots)\ , \\ +\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

there being n lines and n terms in each line. Cayley would have represented this later, in 1855, by the notation

$$\begin{array}{l} (\alpha\ \beta\ \ldots\ldots \between \xi,\ \eta \between x,\ y,\ \ldots\ldots) \\ \left| \begin{array}{l} \alpha'\beta'\ \ldots\ldots \\ \ldots\ldots\ldots\ldots \end{array} \right| \end{array}$$

and called a "bipartite." A still later notation, as Muir[3] points out, is

$$\begin{array}{ccc|c} \xi & \eta & \ldots\ldots & \\ \hline \alpha & \beta & \ldots\ldots & x \\ \alpha' & \beta' & \ldots\ldots & y \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \end{array}$$

from which each term of the final expansion is very readily obtained by multiplying an element, β' say, of the square array, by the two elements (y, η) which lie in the same row and column with it but out-

[1] A. Cayley, *Cambridge Math. Journal*, Vol. IV, p. 119–20; *Collected Math. Papers*, Vol. I, p. 55–62; Th. Muir, *op. cit.*, Vol. II, p. 15.

[2] A. Cayley, *Trans. Cambridge Philos. Soc.*, Vol. VIII (1843), p. 1–16; *Collected Math. Papers*, Vol. I, p. 63–79.

[3] Thomas Muir, *op. cit.*, Vol. II, p. 18, 19.

side the array. The three determinants which are viewed as "derivational functions" of this function U are denoted by Cayley KU, FU, $\mathfrak{F}U$, symbols which "possess properties which it is the object of this section to investigate," KU being what later came to be called the "discriminant" of U.

In the second part of this paper of 1843, Cayley deals (as Muir explains)[1] with a class of functions obtainable from the use of m sets of n indices in the way in which a determinant is obtainable from only two sets. The general symbol used for such a function is

$$\left\{\begin{matrix} A_{\rho_1} & \sigma_1 & \tau_1 \ldots \\ \rho_2 & \sigma_2 & \tau_2 \ldots \\ \ldots & \ldots & \ldots \\ \rho_n & \sigma_n & \tau_n \ldots \end{matrix}\right\}$$

which stands for the sum of all the different terms of the form $\pm_r \cdot \pm_s \cdot \pm_t \ldots A_{\rho_{r_1}\sigma_{s_1}\tau_{t_1}} \ldots \times \ldots \times A_{\rho_{r_n}\sigma_{s_n}\tau_{t_n}} \ldots$ where $r_1, r_2, \ldots, r_n$; $s_1, s_2, \ldots, s_n$; $t_1, t_2, \ldots, t_n$; $\ldots$ denote any permutation, the same or different, of the series $1, 2, \ldots, n$, and where $\pm_r$ denotes $+$ or $-$, according as the number of inversions in $r_1, r_2, \ldots, r_n$ is even or odd. Using a $\dagger$ to denote that the ρ's are unpermutable, he obtains

$$\left\{\begin{matrix} A\rho_1\,\sigma_1 \ldots \\ \ldots\ldots \\ \rho_n\,\sigma_n \ldots \end{matrix}\right\} = 1.2 \ldots m \left\{\begin{matrix} \dagger \\ A\rho_1\,\sigma_1 \ldots \\ \ldots\ldots \\ \rho_n\,\sigma_n \ldots \end{matrix}\right\}$$

when the number of columns, m, is even.

In 1845 Cayley[2] gave the notation which may be illustrated by

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}\right\| = \left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix}\right\|,$$

which represents six equations,

$$\left|\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix}\right| = \left|\begin{matrix} x_1 & x_2 \\ y_1 & y_2 \end{matrix}\right|, \qquad \left|\begin{matrix} a_1 & a_3 \\ b_1 & b_3 \end{matrix}\right| = \left|\begin{matrix} x_1 & x_3 \\ y_1 & y_3 \end{matrix}\right|, \text{ etc.}$$

This is a more general mode of using the double vertical lines which had been first introduced by Cayley in 1843.

[1] Th. Muir, *op. cit.*, Vol. II, p. 63, 64; *Collected Math. Papers*, Vol. I, p. 76.

[2] A. Cayley, *Cambridge Math. Journal*, Vol. IV (1845), p. 193–209; *Collected Math. Papers*, Vol. I, p. 80–94; Th. Muir, *op. cit.*, Vol. II, p. 33.

In 1851, J. J. Sylvester[1] introduced what he calls "a most powerful, because natural, method of notation." He says: "My method consists in expressing the same quantities biliterally as below:

$$\begin{array}{lll} a_1\alpha_1 & a_1\alpha_2 \ldots\ldots & a_1\alpha_n \\ a_2\alpha_1 & a_2\alpha_2 \ldots\ldots & a_r\alpha_n \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot \\ a_n\alpha_1 & a_n\alpha_2 \ldots\ldots & a_n\alpha_n \end{array}$$

where, of course, whenever desirable, instead of $a_1, a_2, \ldots a_n$, and $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$, we may write simply $a, b, \ldots\ldots, l$ and $\alpha, \beta, \ldots\ldots, \lambda$, respectively. Each quantity is now represented by two letters; the letters themselves, taken separately, being symbols neither of quantity nor of operation, but mere umbrae or ideal elements of quantitative symbols. We have now a means of representing the determinant above given in compact form; for this purpose we need but to write one set of umbrae over the other as follows:

$$\begin{pmatrix} a_1 & a_2 \ldots\ldots a_n \\ \alpha_1 & \alpha_2 \ldots\ldots \alpha_n \end{pmatrix}.$$

If we now wish to obtain the algebraic value of this determinant, it is only necessary to take $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ in all its $1.2.3 \ldots\ldots n$ different positions, and we shall have

$$\left\{ \begin{matrix} a_1 & a_2 \ldots\ldots a_n \\ \alpha_1 & \alpha_2 \ldots\ldots \alpha_n \end{matrix} \right\} = \sum \pm \{a_1\alpha_{\theta_1} \times a_2\alpha_{\theta_2} \times \ldots\ldots \times a_n\alpha_{\theta_n}\},$$

in which expression $\theta_1, \theta_2, \ldots\ldots, \theta_n$ represents some order of the numbers $1, 2, \ldots\ldots, n$, and the positive or negative sign is to be taken according to the well-known dichotomous law." An extension to "compound" determinants is also indicated. Since $\left\{ \begin{matrix} a & b \\ \alpha & \beta \end{matrix} \right\}$ denotes $a\alpha \cdot b\beta - a\beta \cdot b\alpha$, Sylvester lets $\left\{ \begin{matrix} \overline{ab} & \overline{cd} \\ \alpha\beta & \gamma\delta \end{matrix} \right\}$ denote

$$\begin{matrix} ab \\ \alpha\beta \end{matrix} \times \begin{matrix} cd \\ \gamma\delta \end{matrix} - \begin{matrix} ab \\ \gamma\delta \end{matrix} \times \begin{matrix} cd \\ \alpha\beta \end{matrix},$$

"that is,

$$\left\{ \begin{matrix} (a\alpha \times b\beta) \\ -(a\beta \times b\alpha) \end{matrix} \right\} \times \left\{ \begin{matrix} (c\gamma \times d\delta) \\ -(c\delta \times d\gamma) \end{matrix} \right\} - \left\{ \begin{matrix} (a\gamma \times b\delta) \\ -(a\delta \times b\gamma) \end{matrix} \right\} \times \left\{ \begin{matrix} (c\alpha \times d\beta) \\ -(c\beta \times d\alpha) \end{matrix} \right\},$$

[1] J. J. Sylvester, *Phil. Magazine* (4), Vol. I (1851), p. 295–305; *Collected Math. Papers*, Vol. I, p. 241; Th. Muir, *op. cit.*, Vol. II, p. 58, 59.

and in general the compound determinant

$$\left\{\begin{matrix}\overline{a_1 \quad b_1 \ldots\ldots l_1} & \overline{a_2 \quad b_2 \ldots\ldots l_2} & \ldots\ldots & \overline{a_r \quad b_r \quad l_r} \\ \alpha_1 \quad \beta_1 \ldots\ldots \lambda_1 & \alpha_2 \quad \beta_2 \ldots\ldots \lambda_2 & \ldots\ldots & \alpha_r \quad \beta_r \quad \lambda_r\end{matrix}\right\}$$

will denote

$$\sum \pm \left\{\begin{matrix} a_1 & b_1 \ldots\ldots l_1 \\ \alpha_{\theta_1} & \beta_{\theta_1} \ldots\ldots \lambda_{\theta_1}\end{matrix}\right\} \times \left\{\begin{matrix} a_2 & b_2 \ldots\ldots l_2 \\ \alpha_{\theta_2} & \beta_{\theta_2} \ldots\ldots \lambda_{\theta_2}\end{matrix}\right\} \times \ldots\ldots \times \left\{\begin{matrix} a_r & b_r \ldots\ldots l_r \\ \alpha_{\theta_r} & \beta_{\theta_r} \ldots\ldots \lambda_{\theta_r}\end{matrix}\right\},$$

where, as before, we have the disjunctive equation $\theta_1, \theta_2, \ldots\ldots, \theta_r = 1, 2, \ldots\ldots, r$."

Sylvester gives a slightly different version of his umbral notation in 1852: "If we combine each of the n letters $a, b, \ldots\ldots, l$ with each of the other n, $\alpha, \beta, \ldots\ldots, \lambda$, we obtain n^2 combinations which may be used to denote the terms of a determinant of n lines and columns, as thus:

$$\begin{matrix} a\alpha, & a\beta & \ldots\ldots & a\lambda \\ b\alpha, & b\beta & \ldots\ldots & b\lambda \\ \cdot & \cdot & \cdot & \cdot \\ l\alpha & l\beta & \ldots\ldots & l\lambda \end{matrix}.$$

It must be well understood that the single letters of either set are mere umbrae, or shadows of quantities, and only acquire a real signification when one letter of one set is combined with one of the other set. Instead of the inconvenient form above written, we may denote the determinant more simply by the matrix

$$\begin{matrix} a, & b, & c \ldots\ldots l \\ \alpha, & \beta, & \gamma \ldots\ldots \lambda. \end{matrix}$$"[1]

In place of Sylvester's umbral notation $\left\{\begin{matrix} a_1\, a_2 \ldots\ldots a_n \\ b_1\, b_2 \ldots\ldots b_n \end{matrix}\right\}$ Bruno[2] writes $\sum(\pm a_1^{\phi_1} a_2^{\phi_2} \ldots\ldots a_p^{\phi_p} \ldots\ldots a_n^{\phi_n}\}$.

Following Sylvester in matters of notation, Reiss[3] indicates the expansion of a five-line determinant in terms of the minors formed

[1] J. J. Sylvester, *Cambridge and Dublin Mathematical Journal*, Vol. VII (Cambridge, 1852), p. 75, 76; *Collected Mathematical Papers*, Vol. I (1904), p. 305.

[2] F. Faà de Bruno, *Liouville Journal de Math.* (1), Vol. XVII (1852), p. 190–92; Th. Muir, *op. cit.*, Vol. II, p. 72.

[3] M. Reiss, *Beiträge zur Theorie der Determinanten* (Leipzig, 1867); Th. Muir, *op. cit.*, Vol. III (1920), p. 21, 22.

from the second and fourth rows and the minors formed from the first, third, and fifth rows, by

$$\mathfrak{si}\begin{pmatrix}2&4&1&3&5\\1&2&3&4&5\end{pmatrix}\sum_{1\,2\,3\,4\,5}^{\beta_1\beta_2}\epsilon\begin{pmatrix}2&4\\\beta_1&\beta_2\end{pmatrix}\begin{pmatrix}1&3&5\\\beta_3&\beta_4&\beta_5\end{pmatrix},$$

where, firstly, $\begin{pmatrix}2&4\\\beta_1&\beta_2\end{pmatrix}\begin{pmatrix}1&3&5\\\beta_3&\beta_4&\beta_5\end{pmatrix}$ is the typical product of two complementary minors; secondly, the accessories to the Σ imply that for β_1, β_2 are to be taken any two integers from 1, 2, 3, 4, 5; and thirdly, the remaining items are connected with the determination of sign.

A somewhat uncommon symbol is $\overline{r}\,\underline{s}$, used by Dodgson[1] to designate the element in the (r, s)th place.

A somewhat clumsy notation due to E. Schering[2] is illustrated by the following: "A member of the determinant can accordingly always be represented by

$$\prod_{\nu=1}^{\nu=n} E_{\eta_\nu} k_\nu \times \mathfrak{Z} \prod_{m,\mu} (\eta_m-\eta_\mu)(k_m-k_\mu) \times \mathfrak{Z} \prod_{a,\alpha} (h_a-h_\alpha)(a-\alpha)$$
$$\times \mathfrak{Z} \prod_{b,\beta} (k_b-k_\beta)(b-\beta)\ ,\text{"}$$

where E is any element denuded of its compound suffix, and $\mathfrak{Z}$ is a function-symbol corresponding to Scherk's ϕ, and being used such that $\mathfrak{Z}(x)$ is equal to $+1$ or 0 or -1 according as x is greater than, equal to, or less than 0.

Cayley's notation of a matrix[3] is exemplified by

$$\left|\begin{array}{llll}\alpha, & \beta, & \gamma & \ldots\\ \alpha', & \beta', & \gamma' & \ldots\\ \alpha'', & \beta'', & \gamma'' & \ldots\\ \cdot & \cdot & \cdot & \cdot\end{array}\right|,$$

a matrix being defined as a system of quantities arranged in the form of a square, but otherwise quite independent. Cayley says that the equations

$$\left.\begin{array}{lll}\xi=\alpha x & +\beta y & +\gamma z\ \ldots\\ \eta=\alpha' x & +\beta' y & +\gamma' z\ \ldots\\ \zeta=\alpha'' x & +\beta'' y & +\gamma'' z\ \ldots\end{array}\right\}$$

[1] C. L. Dodgson, *Elementary Theory of Determinants* (London, 1867); Th. Muir, *op. cit.*, Vol. III, p. 24.

[2] E. Schering, *Abhandlungen d. K. Gesellsch. d. Wissensch.* (Göttingen), Vol. XXII (1907); Th. Muir, *op. cit.*, Vol. III, p. 71.

[3] A. Cayley, *Crelle's Journal*, Vol. 50 (1855), p. 282–85; *Collected Math. Papers*, Vol. II, p. 185–88; Muir, *op. cit.*, Vol. II, p. 85.

may be written[1]

$$(\xi, \eta, \zeta \ldots .) = \overbrace{\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots . \\ \alpha', & \beta', & \gamma', & \ldots . \\ \alpha'', & \beta'', & \gamma'', & \ldots . \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \cdot \end{vmatrix}}(x, y, z \ldots .)$$

and the solution of the equations in the matrix form

$$(x, y, z \ldots .) = \overbrace{\begin{vmatrix} \alpha, & \beta, & \gamma & \ldots . \\ \alpha', & \beta', & \gamma' & \ldots . \\ \alpha'', & \beta'', & \gamma'' & \ldots . \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \cdot \end{vmatrix}^{-1}}(\xi, \eta, \zeta \ldots .)$$

where

$$\begin{vmatrix} \alpha, & \beta, & \gamma & \ldots . \\ \alpha', & \beta', & \gamma' & \ldots . \\ \alpha'', & \beta'', & \gamma'' & \ldots . \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \cdot \end{vmatrix}^{-1}$$

stands for the inverse matrix.

463. *Compressed notations.*—H. J. S. Smith,[2] when the elements are double suffixed, places between two upright lines a single element with variable suffixes and then appends an indication of the extent of the variability. For example, he writes

$$\sum\left(\pm\frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2}\cdots\frac{\partial x_n}{\partial y_n}\right)$$

in the form

$$\left|\frac{dx_\alpha}{dy_\beta}\right|, \qquad \left.\begin{matrix}\alpha=1, 2, \ldots , n \\ \beta=1, 2, \ldots , n\end{matrix}\right\} .$$

In 1866 L. Kronecker[3] made similar use of $|b_{ik}|$; he used also

$$|a_{h1}, a_{h2}, \ldots , a_{hn}| , \quad (h=1, 2, \ldots , n) .$$

[1] In the *Collected Math. Papers*, Vol. II, p. 185, parentheses are placed outside of the vertical lines, on the first row, which were not given in 1855.

[2] H. J. S. Smith, *Report British Assoc.*, Vol. XXXII (1862), p. 503–26; *Collected Math. Papers*, Vol. I (1894), p. 230.

[3] L. Kronecker in *Crelle's Journal*, Vol. LXVIII (1868), p. 276; *Monatsberichte d. Akademie* (Berlin, 1866), p. 600; *Werke*, Vol. I (1895), p. 149, 150.

Salmon[1] ordinarily uses Cayley's two vertical bars, but often "for brevity" writes $(a_1, b_2, c_3 \ldots\ldots)$ where $a_1, b_2, c_3, \ldots\ldots$, are elements along the principal diagonal, which resembles designations used by Bézout and Jacobi.

E. H. Moore[2] makes the remark of fundamental import that a determinant of order t is uniquely defined by the unique definition of its t^2 elements in the form a_{uv}, where the suffixes uv run independently over any (the same) set of t distinct marks *of any description whatever.* Accordingly, in determinants of special forms, it is convenient to introduce in place of the ordinary $1, 2, \ldots\ldots, t$ some other set of t marks. Thus, if one uses the set of t bipartite marks

$$gj \qquad \begin{pmatrix} g=1, \ldots\ldots, m \\ j=1, \ldots\ldots, n \end{pmatrix}$$

and denotes a_{uv} by a_{fihk}, the determinant $A = |a_{fihk}|$ of order mn, where throughout

$$a_{fihk} = b_{fh}^{(i)} \cdot c_{ik}^{(h)} \qquad \begin{pmatrix} f,\ h=1, \ldots\ldots, m \\ i,\ k=1, \ldots\ldots, n \end{pmatrix},$$

is the product of the n determinants $B^{(i)}$ of order m and the m determinants $C^{(h)}$ of the order n:

$$A = B^{(1)} \ldots\ldots B^{(n)} \cdot C^{(1)} \ldots\ldots C^{(m)}, \text{ where } B^{(i)} = \left| b_{fh}^{(i)} \right|, C^{(h)} = \left| c_{ik}^{(h)} \right|.$$

Kronecker[3] introduced a symbol in his development of determinants which has become known as "Kronecker's symbol," viz., δ_{ik}, where $\begin{matrix} i=1, 2, \ldots\ldots, m \\ k=1, 2, \ldots\ldots, n \end{matrix}$, and $\delta_{ik}=0$ when $i \gtrless k$, $\delta_{kk}=1$. The usual notation is now δ_s^r; Murnaghan writes $\left[{r \atop s}\right]$. A generalization of the ordinary "Kronecker symbol" was written by Murnaghan in 1924[4] in the form $\delta^{r_1, \ldots\ldots, r_m}_{s_1, \ldots\ldots, s_m}$ ($m \leq n$, in space of n dimensions), and in 1925[5] in the form $\begin{bmatrix} r_1\, r_2 \ldots\ldots r_m \\ s_1\, s_2 \ldots\ldots s_m \end{bmatrix}$, and appears in the outer multiplication of tensors.

[1] George Salmon, *Modern Higher Algebra* (Dublin, 1859, 3d. ed., Dublin, 1876), p. 1.

[2] E. H. Moore, *Annals of Mathematics* (2d ser.), Vol. I (1900), p. 179, 180.

[3] Leopold Kronecker, *Vorlesungen über die Theorie der Determinanten*, bearbeitet von Kurt Hensel, Vol. I (Leipzig, 1903), p. 316, 328, 349.

[4] F. D. Murnaghan, *International Mathematical Congress* (Toronto, 1924); Abstracts, p. 7.

[5] F. D. Murnaghan, *American Math. Monthly*, Vol. XXXII (1925), p. 234.

Sheppard[1] follows Kronecker in denoting by $|d_{qr}|$ and $|d_{rq}|$ the determinants whose elements in the qth column and rth row are respectively d_{qr} and d_{rq}. The notation of summation

$$\begin{pmatrix} p=1, 2, \ldots, m; \\ q=1, 2, \ldots, m \end{pmatrix} \sum_{s=1}^{m} D_{ps}d_{qs} = \begin{cases} D \text{ if } q=p \\ 0 \text{ if } q \neq p \end{cases},$$

where D_{ps} (cofactor of d_{ps} in the determinant D) is simplified by dropping the sign of summation and using Greek letters;[2] thus

$$\begin{pmatrix} p=1, 2, \ldots, m; \\ q=1, 2, \ldots, m \end{pmatrix} d^{p\sigma}d_{q\sigma} = \begin{cases} 1 \text{ if } q=p \\ 0 \text{ if } q \neq p \end{cases},$$

where $d^{p\sigma} \equiv$ (cofactor of $d_{p\sigma}$ in D) $\div D$. This, in turn,[3] is condensed into $|^{p}_{q}$; also $d^{\lambda\sigma}d_{\mu\sigma} = |^{\lambda}_{\mu}$ ('unit $\lambda\mu$'). The symbol adopted by A. Einstein is δ^{λ}_{μ}, while J. E. Wright, in his *Invariants of Quadratic Differential Forms*, uses $\eta_{\lambda\mu}$. Says Sheppard: "We can treat the statement $d^{\lambda\sigma}d_{\mu\sigma} = |^{\lambda}_{\mu}$ as a set of equations giving the values of $d^{\lambda\sigma}$ in terms of those of $d_{\lambda\sigma}$. If, for instance, $m=20$, the set $d_{\lambda\sigma}$ contains 400 elements,"[4] and this expression is a condensed statement of the 400 equations (each with 20 terms on one side) which give the 400 values of $d_{\lambda\sigma}$.

464. *The Jacobian.*—The functional determinant of Jacobi,[5] called the "Jacobian," is presented by Jacobi, himself, in the following manner: "Propositis variabilium $x, x_1, \ldots, x_n$ functionibus totidem $f, f_1, \ldots, f_n$, formentur omnium differentialia partialia omnium variabilium respectu sumta, unde prodeunt $(n+1)^2$ quantitates, $\frac{\partial f_i}{\partial x_k}$. Determinans $\sum \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \ldots \frac{\partial f_n}{\partial x_n}$, voco Determinans functionale." It was represented by Donkin[6] in the form $\frac{\partial(f_1, f_2, \ldots, f_m)}{\partial(z_1, z_2, \ldots, z_m)}$, by Gordan[7] $\begin{pmatrix} f_1, \ldots, f_m \\ z_1, \ldots, z_m \end{pmatrix}$, by L. O. Hesse $(f_1, f_2, \ldots, f_m)$, and by Weld[8] $J(f_1, f_2, \ldots, f_m)$.

[1] W. F. Sheppard, *From Determinant to Tensor* (Oxford, 1923), p. 39.

[2] *Op. cit.*, p. 41, 47, 49. [3] *Op. cit.*, p. 50. [4] *Op. cit.*, p. 57.

[5] C. G. J. Jacobi, *Crelle's Journal*, Vol. XXII (1841), p. 327, 328; *Werke*, Vol. III, p. 395. See also *Encyclopédie des scien. math.*, Tome I, Vol. II (1907), p. 169.

[6] W. F. Donkin, *Philosophical Transactions* (London), Vol. CXLIV (1854), p. 72.

[7] P. Gordan, *Vorlesungen über Invariantentheorie* (Leipzig, 1885), Vol. I, p. 121.

[8] L. G. Weld, *Theory of Determinants* (New York, 1896), p. 195.

465. *Hessian.*—If u be a function of $x_1, x_2, \ldots, x_n$ and $y_1, \ldots, y_n$ its differential coefficients with respect to these variables, the Jacobian of $y_1, \ldots, y_n$ is called the Hessian of u and is denoted[1] by $H(u)$ so that $H(u) = |u_{ik}|$, where u_{ik} is $\frac{dy_i}{dx_k}$.

466. *Cubic determinants.*—The set of elements in the leading diagonal may be written $a_{111}, a_{222}, \ldots, a_{nnn}$. The cubic determinant may be denoted by[2] $\Sigma \pm a_{111}a_{222} \ldots a_{nnn}$, or by $|a_{ijk}|$ (i. j, $k=1$, 2 $\ldots\ldots$, n). "The multiple-suffix notation is not very attractive," says Muir.[3]

467. *Infinite determinants.*—Kötterlitzsch[4] solved in 1870 a system of an infinity of linear equations which he writes

$$\left.\begin{matrix} 0 \\ m \\ \infty \end{matrix}\right| \sum_0^\infty {}^p f_0(m, p)x_p = \phi m \,.$$

The determinant of this system he represents by

$$R = \sum \pm f_0(0, 0)f_0(1, 1)f_0(2, 2) \ldots f_0(\text{p}, \text{p}) \,.$$

468. *Matrix notations.*—Cayley introduced two double vertical lines to denote a matrix in 1843 and 1845, as already shown. It was used by H. J. S. Smith.[5] Whitworth[6] uses in one place triple vertical lines to indicate that three determinant equations may be independently formed from the matrix.

The notation[7] $[a](x)$ for the product of the matrix $[a_{ik}]$ by the complex quantity (x) has been used by Peano and Bôcher; V. Volterra employs the equation between matrices, $\left[\frac{dx_{ik}}{dx}\right] = [a_{ik}][x_{ik}]$, and notations even more condensed, such as the fundamental differential operation Dx defined by the formula

$$D_x[x_{ik}] = [x_{ik}]^{-1}\left[\frac{dx_{ik}}{dx}\right] \,.$$

[1] R. F. Scott, *Treatise on the Theory of Determinants* (Cambridge, 1880), p. 142, 143.

[2] R. B. Scott, *op. cit.*, p. 89, 90; M. Noether, *Mathematische Annalen*, Vol. XVI (1880), p. 551–55.

[3] Th. Muir, *op. cit.*, Vol. III, p. 392.

[4] Th. Kötterlitzsch in *Zeitschrift f. Math. u. Physik* (Leipzig), Vol. XV (1870), p. 231.

[5] H. J. S. Smith, *Mathematical Papers*, Vol. I, p. 374.

[6] W. A. Whitworth, *Trilinear Coordinates* (Cambridge, 1866), p. xxv.

[7] See E. Vessiot in *Encyclopédie des scien. math.*, Tom. II, Vol. III (1910), p. 129.

The notation for fractional matrices,

$$\left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{matrix} \right\|,$$

has been used by E. H. Moore and M. Bôcher[1] to indicate that one may consider two matrices as equal whenever the elements of one matrix can be obtained from the elements of the other matrix by multiplying each element of the first, or each element of the second, by the same quantity not zero.

Besides the double vertical lines of Cayley, there are several other notations for matrices. Round parentheses have been used by many, for instance, by M. Bôcher.[2] G. Kowalewski[3] finds it convenient sometimes to use the double vertical lines, sometimes the round parentheses, and also often a single brace, placed to the left of the elements, thus:

$$\begin{cases} x_{11}, \ x_{12}, \ \ldots\ldots, \ x_{1n}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{m1}, \ \ x_{m2}, \ \cdot \qquad , \ x_{mn} . \end{cases}$$

A variety of new notations for matrices are found in the work of Cullis.[4] A rectangular matrix is indicated as follows:

$$A=[a]_m^n=\begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix}, \qquad [abc]_{1234}=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$$

A', conjugate to A, is denoted thus:

$$A'=\overline{\underline{a}}\,{}_n^m=\begin{bmatrix} a_{11} & a_{21} \ldots\ldots a_{m1} \\ a_{12} & a_{22} \ldots\ldots a_{m2} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{1n} & a_{2n} \ldots\ldots a_{mn} \end{bmatrix}.$$

[1] M. Bôcher, *Introduction to Higher Algebra* (New York, 1907), p. 86. See "Tribune publique," 19, No. 430, *Encyclopédie des scien. math.*, Tom. II, Vol. V (1912).

[2] Maxine Bôcher, *Introduction to Higher Algebra* (New York, 1919), p. 20.

[3] G. Kowalewski, *Determinantentheorie* (1909), p. 51, 55, 66, 214.

[4] C. E. Cullis, *Matrices and Determinoids* (Cambridge), Vol. I (1913), p. 1–12.

Accordingly, $[a]^n_m$, $\overline{a}^m_n$ are conjugate (abbreviations). More general double-suffix notations are:

$$[a_{pq}]^n_m\,,\ \overline{a_{pq}}^m_n\,,\ [a_{p1}]^n_m = \begin{bmatrix} a_{p_11} & a_{p_12} & \cdots & a_{p_1n} \\ a_{p_21} & a_{p_22} & \cdots & a_{p_2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p_m1} & a_{p_m2} & \cdots & a_{p_mn} \end{bmatrix},$$

and its conjugate

$$A' = \overline{a_{p1}}^m_n = \begin{bmatrix} a_{p_11} & a_{p_21} & \cdots & a_{p_m1} \\ a_{p_12} & a_{p_22} & \cdots & a_{p_m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{p_1n} & a_{p_2n} & \cdots & a_{p_mn} \end{bmatrix}.$$

Most general double-suffix notation:

$$A = \begin{bmatrix} uv \ldots\ldots w \\ a \\ pq \ldots\ldots r \end{bmatrix} = \begin{bmatrix} a_{pu} & a_{pv} & \cdots & a_{pw} \\ a_{qu} & a_{qv} & \cdots & a_{qw} \\ \cdot & \cdot & \cdots & \cdot \\ a_{ru} & a_{rv} & \cdots & a_{rw} \end{bmatrix},$$

$$A' = \overline{\underline{\begin{matrix} pq \ldots\ldots r \\ a \\ uv \ldots\ldots w \end{matrix}}} = \begin{bmatrix} a_{pu} & a_{qu} & \cdots & a_{ru} \\ a_{pv} & a_{qv} & \cdots & a_{rv} \\ \cdot & \cdot & \cdots & \cdot \\ a_{pw} & a_{qw} & \cdots & a_{rw} \end{bmatrix}.$$

Double-suffix notation for augmented matrices:

$$[a,b]^{n,\,r}_m = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1r} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & \cdots & b_{2r} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} & \cdots & b_{mr} \end{bmatrix},$$

$$\begin{bmatrix} a \\ b \end{bmatrix}^n_{m,\,r} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}.$$

The conjugate matrices of $[a,\,b]^{n,\,r}_m$ and $\begin{bmatrix} a \\ b \end{bmatrix}^n_{m,\,r}$ are denoted

$$\overline{\underline{\begin{matrix} a \\ b \end{matrix}}}^m_{n,\,r} \qquad \text{and} \qquad \overline{\underline{a,\,b}}^{m,\,r}_n\,.$$

Most general single-suffix notation for matrix and its conjugate:

$$[ab \ldots . k]_{\alpha\beta \ldots . k}\,, \quad \begin{bmatrix} a \\ b \\ \vdots \\ k \end{bmatrix}_{\alpha\beta \ldots . k} .$$

Corresponding to the notations for a matrix of orders m and n, Cullis uses symbols to represent determinoids of orders m and n,

$$(a)^n_m\,,\ (a_{pq})^n_m\,,\ (a_{p1})^n_m\,,\ (a_{1q})^n_m\,,\ \begin{pmatrix} uv \ldots . w \\ a \\ pq \ldots . r \end{pmatrix},$$

$$(abc \ldots . k)_{123 \ldots . m}\,,\ (abc \ldots . k)_{\alpha\beta\gamma \ldots . k}\,.$$

The determinoid of any matrix A is defined as the algebraic sum of all its complete derived products, when each product is provided with a positive or negative sign in accordance with a fixed *rule of signs.*

For "matrix product" Noether[1] employed the symbol ($|$), as in $(S^\rho|p_\rho)$; Weitzenböck[2] changed the notation for the series S^ρ by writing $(a'_\rho|p_\rho)$.

SIGNS FOR LOGARITHMS

469. *Abbreviations for "logarithm."*—The use by some authors of l to represent *latus* and signifying either "root" or "unknown quantity," did not interfere with its introduction as an abbreviation for "logarithm." Two or three abbreviations occur early, namely, "l" and "log.," also "lg." John Napier coined the word "logarithm," but did not use any abbreviation for it in his writings. Kepler,[3] in 1624, used the contraction "Log." So did H. Briggs[4] and W. Oughtred.[5] Oughtred wrote also "Log: Q:" for "Logarithm of the square of." Ursinus[6] employs "L."

[1] Emmy Noether, *Journal f. reine u. angewandte Mathematik*, Vol. CXXXIX (1910), p. 123, 124.

[2] R. Weitzenböck, *Encyklopädie d. math. Wissensch.*, Band III_3, Heft 6 (1922), p. 16.

[3] J. Kepler, *Chilias logarithmorum* (Marpurgi, 1624).

[4] Henry Briggs, *Logarithmicall Arithmetike* (London, 1631), p. 7.

[5] W. Oughtred, *Key of the Mathematicks* (London, 1647), p. 135, 172, and in later editions of his *Clavis mathematicae* (see § 184).

[6] B. Ursinus, *Magnus canon logarithmicus* (Cologne, 1624).

The Italian B. Cavalieri writes in 1632[1] "log." and in 1643[2] "l." Molyneux[3] of Dublin uses "Log. comp." for the complement of the logarithm. When the logarithmic function received the attention of eighteenth-century writers on the calculus, the contractions "l" and "L" became more frequent. W. Jones[4] wrote "L" for "logarithm" and further on "L" for the "log. sine," and "l" for arithmetical complement of "Log. sine." One finds "l" for logarithm in the *Acta eruditorum*,[5] in the textbook of Christian Wolf[6] (who uses also "Log."), in Brook Taylor,[7] in Johann Bernoulli,[8] in Denys Diderot[9] (who employs also "log."). Euler[10] uses "l" in 1728 and 1748. In Sherwin's *Tables* of 1741[11] one finds "Log." and "L.," in P. Varignon[12] "l x." Robert Shirtcliffe[13] writes "$L:A+L:B=L:A\times B$" for $\log A+\log B=\log AB$.

W. Gardiner, in the Introduction to his *Tables* (London, 1742), not only uses "L.," but suggests "L′" to represent the arithmetical complement in common logarithms, so that "$\mathrm{L}.\frac{1}{\mathrm{a}}=\mathrm{L}'a$." Less happy was the suggestion "*cd* log *a*" for the complement of log *a*, made by J. A. Grunert.[14]

1 Bonaventura Cavalieri, *Directorivm generale Vranometricvm* (Bologna, 1632), p. 169.

2 Bonaventura Cavalieri, *Trigonometria* (Bologna, 1643).

3 William Molyneux, *A Treatise of Dioptricks* (London, 1692), p. 92.

4 W. Jones, *Synopsis palmariorum matheseos* (1706), p. 176, 279.

5 *Acta eruditorum* (Leipzig, 1703), p. 451.

6 Christian Wolf, *Elementa matheseos universae*, Vol. I (Halle, 1713), p. 231, 236, 508.

7 Brook Taylor, *Methodus incrementorum.* See also *Phil. Trans.*, Vol. XXX (1717–19), p. 621.

8 Johann Bernoulli in *Histoire de l'académie r. d. sciences*, année 1730 (Paris, 1732), *Mém.*, p. 78.

9 *Memoires sur différens sujets de mathématiques par M. Diderot* (Paris, 1748), p. 42, 43.

10 L. Euler's letter to J. Bernoulli, Dec. 10, 1728 (see G. Eneström in *Bibliotheca mathematica* [3d ser.], Vol. IV [1903], p. 351, 353); Euler, *Introductio in analysin infinitorum* (1748), Lib. I, § 104.

11 *Sherwin's Mathematical Tables* (3d ed.; rev. William Gardiner; London, 1741), p. 16.

12 P. Varignon, *Eclaircissemens sur l'analyse des infiniment petits* (Paris, 1725), p. 116.

13 Robert Shirtcliffe, *The Theory and Practice of Gauging* (London, 1740), p. 31.

14 J. A. Grunert, *Lehrbuch der Mathematik*, Vol. III (Brandenburg, 1832), p. 69. See J. Tropfke, *op. cit.*, Vol. II (2d ed., 1921), p. 210.

The capital letter "L" for logarithm occurs in Thomas Watkins,[1] in Count de Fagnano,[2] also in J. A. Fas,[3] C. F. Fournier.[4]

The sign "$|x$" for log x sometimes occurs;[5] the vertical stroke is probably a modified letter "l."

470. *Different meanings of "log x," "l x," and "L x."*—In more recent time the meanings of "log," "l," and "L" have been differentiated; Leibniz[6] employed "l" when natural logarithms are implied. Cauchy[7] proposed that "l" stand for natural logarithms, and "L" for the logarithms in any system whose base, $b > 1$. Stolz and Gmeiner[8] let "log" stand for the common logarithm of a positive real number, "l" for its natural logarithm, and "L" for the logarithms of a, where a is real or imaginary, though different from zero. Peano[9] writes "log x" for a logarithm to the base e, and "Log x" for one to the base 10. André[10] states that the logarithm to the base e is written "LP," "$\mathcal{L}P$," or "log P."

471. Another notation was proposed by Crelle,[11] in which the base is written above and to the left of the logarithm. This notation is sometimes encountered in more recent books. Thus, Stringham[12] denotes a logarithm to the base b by "${}^b\log$"; he denotes also a natural logarithm by "ln" and a logarithm to the complex modulul k, by "$\log_k$." Stolz and Gmeiner[13] signify by "${}^a\log.\ b$" the "logarithm of b to the base a."

[1] Thomas Watkins in *Philosoph. Transactions*, Vol. XXIX (1714–16), p. 123.

[2] *Opere Mathematiche del Marchese Giulio Carlo De' Toschi di Fagnano*, Vol. III (1912), p. 30.

[3] J. A. Fas, *Inleiding tot de Kennisse en het Gebruyk der Oneindig Kleinen* (Leyden, 1775), p. 29.

[4] C. F. Fournier, *Éléments d'arithmétique et d'algèbre*, Tome II (Nantes, 1842), p. 147.

[5] *Mathematical Gazette*, Vol. XII (1924), p. 61.

[6] *Leibnizens Mathematische Schriften* (ed. C. I. Gerhardt), Vol. I, p. 117.

[7] A. L. Cauchy, *Cours d'analyse* (Paris, 1821), p. 52, 137, 171.

[8] O. Stolz und J. A. Gmeiner, *Theoretische Arithmetik*, Vol. II (Leipzig, 1902), p. 212, 365.

[9] G. Peano, *Lezioni di analisi infinitesimale*, Vol. I (Torino, 1893), p. 33.

[10] Désiré André, *Des Notations mathématiques* (Paris, 1909), p. 64.

[11] A. L. Crelle, *Sammlung Math. Aufsätze 1* (Berlin, 1821), p. 207. See *Encyclopédie des scien. math.*, Tome I, Vol. I (1904), p. 59.

[12] Irving Stringham, *Uniplanar Algebra* (San Francisco, 1893), p. xiii.

[13] O. Stolz und J. A. Gmeiner, *Theoretische Arithmetik*, Vol. II (Leipzig, 1902), p. 211.

Crelle[1] in 1831 and Martin Ohm[2] in 1846 write the base above the "log," thus, "$\overset{c}{\log}\, a$." This symbolism is found in many texts; for instance, in Kambly's *Elementar-Mathematik*.[3] A. Steinhauser[4] wrote "log. nat. z" and "log. brigg. z."

472. *The power of a logarithm.*—The power $(\log x)^m$ is indicated by Reyneau[5] by "$l^m x$," while log (log x) is designated by "l. l x." Spence[6] lets "$L^2\ (x)$" stand for $(\log x)^2$. But more commonly parentheses are used for marking powers; for instance, Carmichael[7] uses the notation $(\log a)^l . (\log b)^m . (\log c)^n$ for powers. Bonnet[8] denotes by "$(ll\ 3)^\alpha$" the expression $[\log (\log 3)]^\alpha$.

Pringsheim[9] marks by "$\lg_k^\nu n$" or by, say, "$\log_2^3 n$," the expression $\{\lg(\lg n)\}^3$, by "$L\kappa\ (x)$" the expression $\lg_0 x . \lg_1 x \ldots . \lg\kappa\ x$.

473. *Iterated logarithms* have received some consideration in the previous paragraph. They will occur again when we consider logarithmic notations which diverge widely from those ordinarily used. We note here the symbolism used by Pringsheim and Molk in their joint *Encyclopédie* article:

$$\text{“}\,{}^2\!\log_b a = \log_b (\log_b a)\ , \ldots ,$$
$$ {}^{k+1}\!\log_b a = \log_b ({}^k\!\log_b a)\ .\text{”}$$ [10]

474. *Marking the characteristic.*—In the earliest logarithmic tables (except those of Bürgi) no fractions at all were used, the numbers being integers. When the point or comma first entered, it was not as a separator of integers from fractions, but as a mode of marking the integers into groups for more convenient and rapid reading. Thus,

[1] A. L. Crelle in *Crelle's Journal*, Vol. VII (1831), p. 327.

[2] M. Ohm, *Der Geist der Differential- and Integral-Rechnung* (Erlangen, 1846), p. 4.

[3] Ludwig Kambly, *Elementar-Mathematik*, 1. Teil, "Algebra" (Breslau, 1906), p. 57.

[4] Anton Steinhauser, *Lehrbuch der Mathematik*, "Algebra" (Vienna, 1875), p. 278, 283.

[5] Charles Reyneau, *Analyse demontrée*, Vol. II (Paris, 1708), p. 802.

[6] William Spence, *Theory of . . . . Logarithmic Transcendents* (London, 1809), p. viii.

[7] R. Carmichael, *Der Operationscalcul* (trans. C. H. Schnuse; Braunschweig, 1857), p. 87.

[8] Ossian Bonnet, *Journal de mathématiques*, Vol. VIII (Paris, 1845), p. 74.

[9] Alfred Pringsheim, *Mathematische Annalen*, Vol. XXXV (1890), p. 317, 333.

[10] A. Pringsheim and J. Molk, *Encyclopédie des scien. math.*, Tom. I, Vol. I (1907), p. 195.

in Napier's *Descriptio* of 1614 one finds, in the tables, the sine of 30°30′ given as "5, 0 7 5, 3 8 4" and its logarithm as "6, 7 8 1, 8 2 7." Here both the natural sine and its logarithm are integers, and the commas serve to facilitate the reading. As late as 1636, in Gunter's logarithms, the comma is used not as a separator of integers and fractions, but as a separator of the last five figures; the logarithm of 20 is given as "1 3 0 1, 0 2 9 9 9."

The use of the characteristic is found in the logarithmic tables of 1617 and 1624, published by Briggs,[1] who separated the characteristic from the mantissa by a comma. He placed also a comma after every fifth digit in the fourteen decimal places. Thus, Briggs writes log 1 6 5 0 3 thus: "4, 2 1 7 5 6, 2 8 9 9 6, 6 9 4 3." In 1624 Briggs writes the decimal "$2\underline{5}$," and its logarithm "0, 3 9 7." In the tables of H. Gellibrand,[2] published by Vlacq, the comma is used in the same manner. The omission of the characteristic in the tables of numbers occurs first in Sherwin (1705).[3]

475. There long existed, in fact there still exists, divergence in the mode of separating the mantissa and characteristic. Beutel[4] in 1690 writes the logarithm of "$160.\frac{88}{100}$" as "2.2065203." Metzburg[5] uses the dot (sometimes the comma) as logarithmic separatrix, and the comma regularly as separatrix in ordinary decimal fractions.

Vega[6] uses regularly the comma as decimal separatrix and also as logarithmic separatrix, the dot being reserved for multiplication. André expresses himself on this matter as follows: "In the logarithmic tables of Lalande the characteristic is followed, not by a comma, but by a point, only the point is placed on the principal line as the comma would be placed. In a book on elementary Arithmetic, the numeral for units is separated from the one for tenths by a kind of hyphen: the number 7, 2 3 1 8 is written 7- 2 3 1 8. . . . . In a certain five-place table of logarithms the mantissa is divided into two parts, separated by an interval in which is placed a point: The mantissa 1 5 4 7 3 appears under the form 15 . 4 7 3. These diverse employments of the

[1] Henry Briggs, *Logarithmorum Chilias prima* (London, 1617); *Arithmetica Logarithmica sive logarithmorum chiliades triginta* (London, 1624).

[2] Henry Gellibrand, *Trigonometria Britannica* (Goudae, 1633).

[3] H. Sherwin, *Mathematical Tables* (London, 1705).

[4] Tobias Beutel, *Geometrischer Lust Garten* (Leipzig, 1690), p. 240, 241.

[5] Des Freyherrn von Metzburg, *Anleitung zur Mathematik*, 1. Theil (Wien, 1798), p. 88, 274; 3. Theil, p. 41.

[6] Georg Freyherr von Vega, *Vorlesungen über die Mathematik*, Band I, 3. Aufl. (Wien, 1802), p. 96, 364.

point constitute so many faults: the point should not enter a numerical expression other than as a contracted sign of multiplication."[1]

476. The logarithms of fractions less than one gave rise to a diversity of notation which has prevailed down to the present time.

Richard Norwood[2] in 1631 (writing "s" when he means "log sine") gives "s, D 6 7 deg. 2 3′ 9,9 6 5 2 4 8 0," where D is an angle of the triangle $A\ D\ B$. Here we have the characteristic increased by 10 and separated from the mantissa by a comma. Similarly, in John Ward,[3] "the Sine of the Angle C 49°−30′ 9,8810455."

The writing of the negative sign over the characteristic, to indicate that the characteristic is negative, is found in Oughtred's *Clavis mathematicae* (except the 1631 edition which does not consider logarithms). We quote from the 1652 edition (p. 122): "Ut numeri 436, Log: est 2,6394865 at numeri 4 3 6 0 0 , est 4, 6 3 9 4 8 6 5. et numeri 4 $\underline{36}$, Log: est 0, 6 3 9 4 8 6 5. Denique numeri 0$\underline{|0\ 0\ 4\ 3\ 6}$. Log: est $\overline{3}$, 6 3 9 4 8 6 5." Observe that the characteristic is separated from the mantissa by a comma, and that a different sign is used for decimal fractions. In his *Circles of Proportion* (London, 1632), page 4, Oughtred gives log 2 as "0. 3 0 1 0 3."

Ozanam[4] in 1692 gives the logarithm of $\frac{3}{4}$ as "−0. 1 2 4 9 3 8 8," the fractional part being here negative and the minus sign placed in front of the logarithm. Likewise, Benito Bails[5] gives "log. $\frac{17}{187}$ = −1, 0 4 1 3 9 3."

477. Oughtred's practice of placing the minus sign of a negative characteristic above the characteristic has found favor among British authors of recent time, such as H. S. Hall and S. R. Knight,[6] E. M. Langley,[7] C. G. Knott,[8] and F. Castle.[9] Langley directs computors "to reject the superfluous 10." The notation "$\overline{2}$. 7 9 5 8 0 0 " is found also in the *Report of the British Association* (1896), page 75. However, it has been far from universal even in England. A. de

[1] Désiré André, *Des Notations mathématiques* (Paris, 1909), p. 20.

[2] Richard Norwood, *Trigonometrie* (London, 1631), Book I, p. 20.

[3] *Posthumous Works of John Ward* (London, 1730), p. 74.

[4] J. Ozanam, *Cours de mathematiques*, Vol. II (new ed.; Paris, 1692), p. 65.

[5] *Principios de matematica de la real academia de San Fernando por Don Benito Bails* (2d. ed.), Vol. I (Madrid, 1788), p. 191.

[6] H. S. Hall and S. R. Knight, *Elementary Algebra*, ed. F. L. Sevenoak (New York, 1895), p. 339.

[7] Edward M. Langley, *A Treatise on Computation* (London, 1895), p. 111, 123.

[8] C. G. Knott, *Logarithmical Tables* (Edinburgh, 1914).

[9] Frank Castle, *Practical Mathematics for Beginners* (London, 1905), p. 127.

Morgan states that "9 is written for $\bar{1}$, that this result may be too great by 10, as are all the logarithms in the table of tangents": there occurs also "19· 8 2 3 9 0 8 8 − 2 0."[1]

On the European continent this notation has been used to a limited extent. Fournier[2] in 1842 wrote "log. 0, 3 4 1 = $\bar{1}$. 5 3 7 5 4." A. Fontaine[3] gives "$\dot{2}$ 3 0 2 5 8 5 0 9 2 9 9 4, etc. = l 10," a designation which did not meet with favor. Some French writers, for example Bézout,[4] and many German, Austrian, and Dutch writers, place the minus sign of a negative characteristic before the logarithm. Segner[5] in 1767 said, "Numeri 0, 5 7 9 3 2 logarithmus est = − 1, 7 6 2 9 1 8 5, in quo signum − ad solam characteristicam refertur," but in the trigonometrical part he writes "log. sin. C = 9, 6 1 6 3 3 8 2."

C. Scherffer[6] writes "Log. sin. 36° = 9, 7 6 9 2 1 8 7" and "Log. $\frac{1}{2}$ = −1, 6 9 8 9 7 0 0." Blassière[7] says: "... pour l'index −1 on écrira ,9; pour −2, on substitue ,8; pour −3 on écrit ,7; et ainsi de suite." "l 0,6 = , 9, 7 7 8 1 5 1 3." The trend of European practice is shown further by the following quotation from Prändel:[8] "Some place the negative characteristic after the mantissa;[9] for example, 0. 8 7 5 0 6 1 3 − 1, and 0. 9 0 3 0 9 0 0 − 3. Others, as Prof. Böhm, write the minus sign over the prefixed characteristic, as in $\bar{3}$. 9 0 3 0 9 0 0. Our designation − 3, + 9 0 3 0 9 0 0 would be the best, except that in addition and subtraction the numbers do not appear in a convenient position relative to the other logarithms. We shall select a middle path and designate such hypothetical logarithms which have a negative characteristic and a positive mantissa thus, −1. 8 7 5 0 6 1 3 +),

[1] *Library of Useful Knowledge, Mathematics*, Vol. I (London, 1847): A. de Morgan, *Examples of the Processes of Arithmetic and Algebra*, p. 54, 55.

[2] C. F. Fournier, *Éléments d'arithmétique et d'algèbre*, Tome II (Nantes, 1842).

[3] *Mémoires donnés à l'académie royale des sciences*, non imprimés dans leur temps. Par M. Fontaine, de cette Académie (Paris, 1764), p. 3.

[4] E. Bézout, *Cours le mathématiques*, Tome I (2d ed.; Paris, 1797), p. 181, 188. He uses also negative mantissas.

[5] I. A. de Segner, *Cursus mathematici*, Part I (Halae, 1767), p. 135, 356.

[6] *Institutiones geometriae sphericae, ... a Carolo Scherffer* (Vienna, 1776), p. 66.

[7] J. J. Blassière, *Institution du Calcul numerique et litteral*, 2. partie (A la Haye, 1770), p. 383.

[8] Johann Georg Prändel, *Algebra* (München, 1795), p. 507, 508.

[9] This notation is found in the following books: J. V. Voigt's *Grundlehren der reinen Mathematik* (Jena, 1791), p. 192; J. F. Lorenz', *Grundlagen der allgemeinen Grössenberechnung* (2d ed.; Helmstadt, 1800), p. 60; A. Steinhauser's *Lehrbuch der Mathematik, Algebra* (Vienna, 1875), p. 278, 283.

$-3.9030900+$. Certainly these can be placed more conveniently one below the other.

"Other mathematicians understand by hypothetical logarithms the complement of the negative logarithms with respect to 10.0000000. For example the hypothetical logarithm of -2.7482944 would be 7.2517056. But computation with this sort of logarithms becomes in the first place too intricate, and in the second place too mechanical."

478. *Marking the last digit.*—Gauss[1] states that von Prasse in his logarithmic *Tafeln* of 1810 placed in different type the last figure in the mantissa, whenever that figure represented an increase in the value of the mantissa. Babbage denoted the increase by a point subscript which the reader scarcely notices, but Lud. Schrön (1860) used a bar subscript which catches the eye at once "and is confusing."[2] Many writers, for instance Chrystal,[3] place a bar above the last digit, to show that the digit has been increased by a unit. F. G. Gausz[4] explains that $\bar{5}$ in the last digit means that it was raised from 4, that $\dot{5}$ means that the remaining decimal has been simply dropped, that a star (✳) prefixed to the last three digits of the mantissa means that the first "difference" given on the line next below should be taken.

479. *Sporadic notations.*—A new algorithm for logarithms was worked out in 1778 by Abel Burja[5], who writes $\frac{a}{b}=m$ where b is the base, a the power (*dignité*), and m the logarithm, so that, expressed in the ordinary notation, $a=b^m$. He finds that $\frac{a}{b}-\frac{c}{d}=\frac{a}{b}-\frac{c^{\frac{b}{d}}}{b}=\frac{a:c^{\frac{b}{d}}}{b}.$

He calls *proportion logarithmique* four quantities of which the second is the same power of the first that the fourth is of the third; he designates it by the sign $\dot{\bowtie}$ as in 2, 8 $\dot{\bowtie}$ 4, 64. Burja proceeds:

"$a^{\overset{n}{\cdot}}$ ou a^n sera la n-ième puissance de a

$a^{\overset{n}{:}}$, la n-ième bipuissance de a ...

[1] K. F. Gauss, *Werke*, Vol. III (1866), p. 242.

[2] J. W. L. Glaisher's "Report on Mathematical Tables" in *Report of British Association* (1873), p. 58.

[3] G. Chrystal, *Algebra*, Part II (Edinburgh, 1889), p. 218.

[4] F. G. Gausz, *Logarithm. und trigonom. Tafeln* (Halle, a. S., 1906), "Erläuterungen."

[5] Abel Burja in *Nouveaux mémoires de l'académie r. d. sciences et des belles lettres*, année 1778 et 1779 (Berlin, 1793), p. 301, 321.

en general,

$a^{\overset{n}{N}}$ sera la n-ième puissance de l'ordre N de a."

The need of the novel symbolism suggested by Burja was not recognized by mathematicians in general. His signs were not used except by F. Murhard[1] of Göttingen who refers to them in 1798. By the sign "$\log \frac{a}{b}$" Robert Grassmann[2] marked the quantity c which yields the relation $b^c = a$. The Germans considered several other notations. In the relation $2^3 = 8$, Rothe in 1811 and Martin Ohm[3] in 1823 suggested "$8 ? 2 = 3$," while Bischoff[4] in 1853 wrote ⋀ for "log," and Köpp[5] in 1860 proposed $\sqrt[8]{2} = 3$. Köpp's sign is favored by Draenert. F. J. Studnička[6] of Prague opposes it as *hässlich* and favors the use of "l" for a natural logarithm, and $\mathfrak{l}$ (derived, as he says, from "log," "lg," $\mathfrak{l}$) for Briggian logarithms, and "$\mathfrak{l}\,\frac{b}{c}$" for logarithms to any base c as the equivalent of Schlömilch's "$\log^c b$." E. Bardey opposes the introduction of any new logarithmic symbol. Paugger[7] simply inverts the radical sign $\sqrt{}$, and writes $\sqrt[m]{p} = a$ and $_a\!\!\!\!\sqrt{}\,p = m$ (identical with $\log_a p = m$). Later on in his *Operationslehre* (p. 81) Paugger uses a modification of the Greek letter λ, as shown in $a^m = b$, $\sqrt[m]{b} = a$, $\lambda_a b = m$. The editor of the *Zeitschrift*, J. C. V. Hoffmann,[8] prefers in the case $a^n = p$, the symbolisms $n = \sqrt[a]{p}$, and for antilogarithm $\sqrt[a]{n} = p$. J. Worpitzky and W. Erler[9] propose a modified l, namely, $\mathfrak{L}$, so that log (bc) to the base a would be written $\underset{a}{\mathfrak{L}}$."

Landen[10] marked by log $(RQ' : PQ')$ the hyperbolic logarithm of $\frac{RQ'}{PQ'}$, or the measure of the ratio of RQ' to PQ.

[1] Friederich Murhard, *System der Elemente der allgemeinen Grössenlehre* (Lemgo, 1798), p. 260.

[2] Robert Grassmann, *Zahlenlehre oder Arithmetik* (Stettin, 1872), p. 45.

[3] See Draenert in *Zeitschr. f. math. u. naturwiss. Unterricht*, Vol. VIII (Leipzig, 1877), p. 266.

[4] Anton Bischoff, *Lehrbuch der Algebra* (Regensburg, 1853), p. 265.

[5] Köpp, *Ausführung gewöhnlicher Zifferrechnungen mittelst Logarithmen* (Osterprogramm des Realgymnasiums zu Eisenach, 1860); also in his *Trigonometrie* (1863) and *Arithmetik* (1864).

[6] *Zeitschr. f. math. u. naturw. Unterricht*, Vol. VIII (1877), p. 403.

[7] *Op. cit.*, Vol. VIII, p. 268, 269.

[8] *Op. cit.*, Vol. VIII, p. 270. [9] *Op. cit.*, Vol. VIII, p. 404.

[10] John Landen, *Mathematical Lucubrations* (London, 1755), Sec. III, p. 93.

Schellbach[1] of Berlin adopted for the operation of *logarithmirung*, in place of the incomplete formula, $\log a=c$, the equation $\underset{b}{\overset{a}{\times}}=c$; he expresses the theorem relating to change of modulus thus,

$$\text{“}\underset{b}{\overset{a}{\times}}\ \underset{k}{\overset{b}{\times}}=\underset{k}{\overset{a}{\times}}.\text{”}$$

As the notation here proposed is not always convenient, he suggests an alternative notation. Just as one has $a\times b$ and $a.b$, $\frac{a}{b}$ and $a:b$, so one may choose $\underset{b}{\overset{a}{\times}}$ and $a\vdots b$. He writes $(a+b)\vdots c$ for $\log (a+b)$ to the modulus c, also $(a\vdots b)\vdots c$ for $\log_c\left(\log \frac{a}{b}\right)$.

480. *Complex numbers.*—Martin Ohm, in his treatment of logarithms of complex numbers, lets "log" represent the infinitely many logarithms of a complex number, "L" the tabular logarithm of the modulus. He states that when the concept of the general power a^x is given the concept of the general logarithm $b\,?\,a$, if by it is meant every expression x such that one has $a^x=\mathrm{b}$ or $e^{x\log a}=b$."[2] If x has an infinite number of values for each value of $\log a$, one sees the reason for the appearance in $a\,?\,b$ of two independent arbitrary constants, as is seen also in the investigations due to J. P. W. Stein, John Graves, and W. R. Hamilton.[3] Ohm says that, since b and a are taken completely general, and a^x has an infinity of values, $b\,?\,a$ is wholly undetermined, unless it is stated for what value of $\log a$ the power a^x is to be taken. He adopts the special notation $b?(a\|\alpha)$, which means the logarithm of b to the base a, when $\log a=\alpha$. He shows that $b?(a\|\alpha)=\frac{\log b}{\alpha}$ is a complete equation. If $a=e=2.718\ldots.$, then Ohm's logarithms reduce to those previously developed by Euler.

De Morgan[4] in developing the general theory of logarithms used "log" for the numerical or tabular value of the logarithm, and let λx

[1] K. H. Schellbach, "Ueber die Zeichen der Mathematik," *Crelle's Journal*, Vol. XII (1834), p. 70–72.

[2] Martin Ohm, *System der Mathematik*, Vol. II (2d ed., 1829), p. 438, 415.

[3] See F. Cajori, *American Mathematical Monthly*, Vol. XX (1913), p. 173–78.

[4] A. de Morgan, *Trans. Cambridge Philos. Soc.*, Vol. VII (1842), p. 186.

stand for "any legitimate solution of $e^{\lambda x}=x$." Later[1] he called λA the "logometer of A" and used also the inverted letter $\curlyvee$, $\curlyvee A$ to mean the line whose logometer is A.

Peano[2] lets $\log x$ represent "la valeur principale du logarithme," and $\log^* x$, "la classe des solutions de l'équation $e^y=x$."

481. *Exponentiation* is the operation of finding x in $a^x=P$. Says André:[3] "It is J. Bourget[4] who, for the first time, pointed out this lacuna and indicated the means of filling it up. He gave the operation of finding the exponent the name exponentiation and represented it by the sign $\div$ which he called 'exponented by.' The equality $64\div 4=3$ is read; 64 exponented by 4 is equal to 3."

482. *Dual logarithms.*—A dual number of the ascending scale is always a continued product of powers of the numbers 1·1, 1·01, 1·001, 1·0001, etc., taken in order, the powers of the numbers alone being expressed. To distinguish these numbers from ordinary numbers the sign $\downarrow$ precedes them. Thus $\downarrow 6, 9, 7, 6=(1{\cdot}1)^6\times(1{\cdot}01)^9\times(1{\cdot}001)^7\times(1{\cdot}0001)^6, \ldots\ldots, \downarrow 6=(1{\cdot}1)^6, \downarrow 0, 6=(1{\cdot}01)^6, \downarrow 0, 0, 6=(1{\cdot}001)^6$. When all but the last digit of a dual number are zeroes, the dual number is called a dual logarithm. Byrne[5] gives also tables of "Dual logarithms and dual numbers of the descending branch of Dual Arithmetic from '0 '0 '0 '1 '0 '0 '0 '0 $\uparrow$ to '3 '6 '9 '9 '0 '0 '0 '0 $\uparrow$ with corresponding natural numbers."

SIGNS IN THEORETICAL ARITHMETIC

483. *Signs for "greater" and "less."*—Harriot's symbols $>$ for greater and $<$ for less (§ 188) were far superior to the corresponding symbols $\overline{\sqsubset}$ and $\underline{\sqsupset}$ used by Oughtred. While Harriot's symbols are symmetric to a horizontal axis and asymmetric only to a vertical, Oughtred's symbols are asymmetric to both axes and therefore harder to remember. Indeed, some confusion in their use occurred in Oughtred's own works, as is shown in the table (§ 183). The first deviation from his original forms is in "Fig. EE" in the Appendix, called the *Horologio*, to his *Clavis*, where in the edition of 1647 there stands $\underline{\sqsubset}$ for $<$, and in the 1652 and 1657 editions there stands $\overline{\sqsubset}$ for $<$. In the text of the *Horologio* in all three editions, Oughtred's regular nota-

[1] A. de Morgan, *Trigonometry and Double Algebra* (London, 1849), p. 130.

[2] G. Peano, *Formulaire mathématique*, Vol. IV (Turin, 1903), p. 229.

[3] Désiré André, *op. cit.*, p. 63.

[4] J. Bourget, *Journal de Mathématiques élémentaires*, Vol. II, p. 12.

[5] Oliver Byrne, *Tables of Dual Logarithms* (London, 1867), p. 7–9. See also Byrne's *Dual Arithmetic* and his *Young Dual Arithmetician.*

tion is adhered to. Isaac Barrow used $\sqsubset$ for "majus" and $\sqsupset$ for "minus" in his *Euclidis Data* (Cambridge, 1657), page 1, and also in his *Euclid's Elements* (London, 1660), Preface, as do also John Kersey,[1] Richard Sault,[2] and Roger Cotes.[3] In one place John Wallis[4] writes $\sqsupset$ for >, $\sqsupset$ for <.

Seth Ward, another pupil of Oughtred, writes in his *In Ismaelis Bullialdi astronomiae philolaicae fundamenta inquisitio brevis* (Oxoniae, 1653), page 1, $\sqsubset$ for "majus" and $\sqsubset$ for "minus." For further notices of discrepancy in the use of these symbols, see *Bibliotheca mathematica*, Volume XII[3] (1911–12), page 64. Harriot's > and < easily won out over Oughtred's notation. Wallis follows Harriot almost exclusively; so do Gibson[5] and Brancker.[6] Richard Rawlinson of Oxford used $\sqsupset$ for greater and $\sqsubset$ for less (§ 193). This notation is used also by Thomas Baker[7] in 1684, while E. Cocker[8] prefers $\sqsubset$ for $\sqsupset$. In the arithmetic of S. Jeake,[9] who gives "$\sqsubset$ greater, $\sqsubset$. next greater, $\sqsupset$. lesser, $\sqsupset$. next lesser, $\sqsubset$ not greater, $\sqsupset$. not lesser, $\sqsubseteq$. equal or less, $\sqsupseteq$. equal or greater," there is close adherence to Oughtred's original symbols.

Ronayne[10] writes in his *Algebra* $\sqsubset$ for "greater than," and $\sqsupset$ for "less than." As late as 1808, S. Webber[11] says: ". . . . . we write $a \sqsubset b$, or $a \mathbin{7} b$; . . . . $a \sqsupset b$, or $a<b$." In Isaac Newton's *De Analysi per Aequationes*, as printed in the *Commercium Epistolicum* of 1712, page 20, there occurs $x \sqsupset \frac{1}{2}$, probably for $x<\frac{1}{2}$; apparently, Newton used here the symbolism of his teacher, I. Barrow, but in Newton's *Opuscula* (Castillion's ed., 1744) and in Lefort's *Commercium Epistolicum* (1856), page 74, the symbol is interpreted as meaning $x>\frac{1}{2}$. Eneström[12]

[1] John Kersey, *Elements of Algebra* (London, 1674), Book IV, p. 177.

[2] Richard Sault, *A New Treatise of Algebra* (London, n.d.).

[3] Roger Cotes, *Harmonia mensurarum* (Cambridge, 1722), p. 115.

[4] John Wallis, *Algebra* (1685), p. 127.

[5] Thomas Gibson, *Syntaxis mathematica* (London, 1655), p. 246.

[6] Thomas Brancker, *Introduction to Algebra* (trans. of Rahn's *Algebra;* London, 1668), p. 76.

[7] Thomas Baker, *Clavis geometrica* (London, 1684), fol. *d* 2 *a*.

[8] Edward Crocker, *Artificial Arithmetick* (London, 1684), p. 278.

[9] Samuel Jeake, Sr., ΛΟΓΙΣΤΙΚΗΛΟΓΙ'Α *or Arithmetick* (London, 1696), p. 12

[10] Philip Ronayne, *Treatise of Algebra* (London, 1727), p. 3.

[11] Samuel Webber, *Mathematics*, Vol. I (Cambridge, Mass., 1808; 2d ed.), p. 233.

[12] G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XII (1911–12), p. 74.

argues that Newton followed his teacher Barrow in the use of ‾‾] and actually took $x<\frac{1}{2}$, as is demanded by the reasoning.

In E. Stone's *New Mathematical Dictionary* (London, 1726), article "Characters," one finds [__ or ‾‾] for "greater" and __] or [‾‾ for "less." In the Italian translation (1800) of the mathematical part of Diderot's *Encyclopédie*, article "Carattere," the symbols are further modified, so that ⊏ and ⌉ stand for "greater than," ⊐ for "less than"; and the remark is added, "but today they are no longer used."

Brook Taylor[1] employed ‾‾] and [‾‾ for "greater" and "less," respectively, while E. Hatton[2] in 1721 used [__ and __], and also > and <. The original symbols of Oughtred are used in Colin Maclaurin's *Algebra*.[3] It is curious that as late as 1821, in an edition of Thomas Simpson's *Elements of Geometry* (London), pages 40, 42, one finds [‾‾ for > and ‾‾] for <.

The inferiority of Oughtred's symbols and the superiority of Harriot's symbols for "greater" and "less" are shown nowhere so strongly as in the confusion which arose in the use of the former and the lack of confusion in employing the latter. The burden cast upon the memory by Oughtred's symbols was even greater than that of double asymmetry; there was difficulty in remembering the distinction between the symbol [‾‾ and the symbol [__. It is not strange that Oughtred's greatest admirers—John Wallis and Isaac Borrow—differed not only from Oughtred, but also from each other, in the use of these symbols. Perhaps nowhere is there another such a fine example of symbols ill chosen and symbols well chosen. Yet even in the case of Harriot's symbolism, there is on record at least one strange instance of perversion. John Frend[4] defined < as "greater than" and > as "less than."

484. *Sporadic symbols for "greater" or "less."*—A symbol constructed on a similar plan to Oughtred's was employed by Leibniz[5] in 1710, namely, "*a* =- significat *a* esse majus quam *b*, et *a* =_ significat *a* esse minus quam *b*." Leibniz borrowed these signs from his teacher Erhard Weigel,[6] who used them in 1693. In the 1749 edition of the *Miscellanea Berolinensia* from which we now quote, these inequality

[1] Brook Taylor, *Phil. Trans.*, Vol. XXX (1717–19), p. 961.

[2] Edward Hatton, *Intire System of Arithmetic* (London, 1721), p. 287.

[3] Colin Maclaurin, *A Treatise of Algebra* (3d ed.; London, 1771).

[4] John Frend, *Principles of Algebra* (London, 1796), p. 3.

[5] *Miscellanea Berolinensia* (Berlin, 1710), p. 158.

[6] *Erhardi Weigelii Philosophia mathematica* (Jenae, 1693), p. 135.

symbolisms are displaced by $>$ and $<$. Other symbolisms which did not meet with favor were Albert Girard's[1] *ff* for *plus que* and § for *moins que*, and Samuel Reyhers'[2] $\dashv$ for $>$, $\vdash$ for $<$, $\Dashv$ for twofold greater ratio or *duplicate* ratio, $\equiv\!|$ for threefold greater ratio or *triplicate* ratio. Hérigone (§ 263) denoted "greater than" by 3|2 and "less than" by 2|3. F. Dulaurens[3] in 1667 adopted $\sqcap$ for "equal," $\ulcorner$ for "greater," and $\urcorner$ for "less." John Bonnycastle[4] introduced in his *Algebra* $\overline{\underline{\diagdown}}$ for $>$ and $\overline{\underline{\diagup}}$ for $<$.

John Wallis[5] used $\overline{\overline{>}}$ for "equal or greater than," and again later[6] the signs $\geqslant$ ("equal or greater"), $\leqslant$ ("equal or less"), and $\overline{\overline{\underline{\diagup}}}$ ("equal, greater, or less"). The first two of these symbols have retained their places in some modern texts, for instance in one of A. R. Forsyth.[7] The third symbol was used by J. F. Lorenz[8] in his *Euklid's Elemente*. The signs $>$ and $<$ slowly worked their way into Continental Europe. The symbols $\geqq$, $\leqq$, which are less compact than those of Wallis, have been attributed to the Frenchman P. Bouguer.[9] In a letter written by Goldbach[10] to Euler, soon after 1734, one reads: "Der Hr. Bouguer brauchet, vor das signum, so Ew. schreiben $\not<$, dieses $\geqq$, welches zwar nicht compendiös, aber sehr expressif ist." This statement is of interest also as indicating that Euler had used $<$ crossed by a stroke, to indicate "not less than." The signs $>$ and $<$ were used by Rahn[11] (§ 194) and by Brancker[12] in his translation of Rahn's book into English (§ 194), also by Johann, Bernoulli,[13] and C. Wolf.[14]

[1] Albert Girard, *Invention nouvelle en l'algebre* (Amsterdam, 1629).

[2] Samuel Reyhers, *Euclides* (Kiel, 1698), Vorrede.

[3] Francisci Dulaurens, *Specimina mathematica duobus libris comprehensa* (Paris, 1667), in list of symbols.

[4] John Bonnycastle, *Introduction to Algebra* (London, 1824), p. 2.

[5] John Wallis, *De sectionibus conicis* (Oxford, 1655), p. 70.

[6] John Wallis in *Philosoph. Trans.*, Vol. V (London, 1670), p. 4011.

[7] A. R. Forsyth, *Theory of Functions of a Complex Variable* (Cambridge, 1893), p. 86.

[8] J. F. Lorenz, *Euklid's Elemente* (ed. C. B. Mollweide; Halle, 1824), p. xxxi.

[9] *Encyclopédie des scien. mathém.*, Tome I, Vol. I (1904), p. 23, n. 123.

[10] P. H. Fuss, *Correspondance mathématique et physique* (St. Peterburg, 1843), Vol. I, p. 304.

[11] Johann Heinrich Rahn, *Teutsche Algebra* (Zürich, 1659), p. 72.

[12] Thomas Brancker, *Introduction to Algebra* (trans. out of the High-Dutch into English; London, 1668), p. 59.

[13] Johann Bernoulli, *Acta eruditorum* (1687), p. 617.

[14] *Elementa matheseos universae*, Tomus I . . . . autore Christiano Wolfo (Halle, 1713), p. 30.

485. *Improvised type.*—How typesetters frequently improvised signs by the use of forms primarily intended for other purposes is illustrated in the inequality signs appearing in 1743 in papers of Euler.[1] The radical sign $\surd$ is turned into the position ⋝ for "greater than" and into ⋜ for "less than." A. G. Kästner[2] in 1758 has the astronomical sign for Aries or Ram to designate "greater than," as shown in 8♈5. Friedrich Schmeisser[3] uses the letter Z, with the slanting stroke much heavier than the rest, as the sign for inequality (*inaequalitas*). Sometimes the sign $\angle$ is used for "less," as in parts of an article by L. Seidel,[4] notwithstanding the fact that this symbol is widely used for "angle."

486. *Modern modifications.*—A notation, the need of which has not been felt by later writers, was introduced in 1832 by E. H. Dirksen[5] of Berlin. He wrote "$n \cdot > Q$" for an infinite series whose terms remain greater than Q "by no assignable positive quantity," and "$n \cdot < Q$" when the terms remain less than Q "by no assignable positive quantity."

Another unusual notation was adopted by W. Bolyai[6] who in 1832 distinguished between $>$, $<$ and $\trianglerighteq$, $\trianglelefteq$. The latter were applied to the absolute values, as in "$-5 <$ et $\trianglerighteq -2$."

Additional symbols are given in the *Algebra* of Oliver, Wait, and Jones:[7] $<$, $>$, $\leqslant$, $\geqslant$, mean "less than," "greater than," "smaller than," "larger than," respectively. Here "larger" and "smaller" take account of the size of the two numbers only; "greater" and "less" have a broader significance, meaning in some cases "higher" and "lower," "later" and "earlier." $\equiv$ means that two numbers are equally large, as "$+1600 \equiv -1600$."

W. E. Byerly[8] used the sign $\prec$ for "equal to or less than" (see also § 682).

[1] L. Euler, *Miscellanea Berolinensia*, Vol. VII (Berlin, 1743), p. 170, 178.

[2] A. G. Kästner, *Anfangsgründe der Arithmetik, Geometrie . . . . Trigonometrie* (Göttingen, 1758), p. 30.

[3] Friedrich Schmeisser, *Lehrbuch der reinen Mathesis*, Erster Theil, *Die Arithmetik* (Berlin, 1817), p. 7.

[4] L. Seidel in *Abhandlungen der Math.-Phys. Classe d. k. Bayerisch. Akad. d. Wissensch.*, Vol. V (München, 1850), p. 386, 387.

[5] E. H. Dirksen, *Abhandlungen der K. P. Akademie d. Wissensch.* (Berlin, 1832), Th. I, p. 82.

[6] Wolfgangi Bolyai de Bolya, *Tentamen* (2d ed.), Tome I (Budapestini, 1897), p. xi.

[7] Oliver, Wait, and Jones, *Treatise on Algebra* (2d ed.; Ithaca, 1887), p. 5, 7.

[8] W. E. Byerly, *Elements of the Integral Calculus* (Boston, 1882), p. 12.

In connection with the theory of limits, Harriot's > and < have been given a curved form by Paul du Bois-Reymond[1] and by A. Pringsheim and J. Molk,[2] thus when $\lim \frac{a_\nu}{b_\nu}$ is zero, they write $a_\nu \prec b_\nu$; when the limit is $+\infty$, $a_\nu \succ b_\nu$.

The sign ≪ was introduced by H. Poincaré and by É. Borel[3] in comparing series like

$$u_0+u_1z+u_2z^2+\cdots\cdot \ll M\left[1+\frac{z}{R^1}+\left(\frac{z}{R^1}\right)^2+\cdots\cdot\right],$$

where the second series has positive coefficients; the modulus of each coefficient of the first series is less than the corresponding coefficient of the second series. The signs ≪ and ≫ are also used for "much less than" and "much greater than."[4]

487. *Signs for absolute difference.*—Oughtred's symbol ∽ for arithmetical or absolute "difference" (§ 184) came to be used widely, for instance, by Kersey,[5] Sault,[6] Jeake,[7] Hatton,[8] Donn,[9] and Cocker.[10] Hérigone[11] lets .∼: stand for "differentia," and ∼ for a minus sign.

There has been great confusion between ∽ and ∼, both in the designation of absolute difference in arithmetic and of similarity in geometry. In place of Oughtred's ∽, Wallis used[12] ∼; he used also[13] $\underset{\sim}{+}$ for plus or absolute difference. In an account of Oughtred's algebra, Wallis[14] gives ∽ for absolute difference; he says, "$a \backsim b$ with me serves indifferently for $a-b$, or $b-a$, according as a or b is the great-

[1] *Annali di matematica pura ed applicata*, Serie II, Vol. IV (1870–71), p. 339.

[2] A. Pringsheim and Molk in *Encyclopédie des scien. math.*, Tom. I, Vol. I (1904), p. 202.

[3] Émile Borel, *Leçons sur les séries divergentes* (Paris, 1901), p. 142.

[4] *Zeitschrift für Physik*, Vol. XIII (1923), p. 366 (the sign was used by H. A. Kramers and W. Pauli); Torsten Heurlinger, *Untersuchungen über die Struktur der Bandenspectra* (Lund, 1918), p. 39.

[5] John Kersey, *Algebra* (London, 1673), p. 3.

[6] Richard Sault, *A New Treatise of Algebra* (London, [n.d.]).

[7] Samuel Jeake, ΛΟΓΙΣΤΙΚΗΛΟΓΙΑ *or Arithmetick* (London, 1696), p. 10–12.

[8] Edward Hatton, *An Intire System of Arithmetick* (London, 1721), p. 287.

[9] Benjamin Donn, *Mathematical Essays* (London, 1758), p. 280.

[10] Edward Cocker, *Artificial Arithmetick*, perused by John Hawkins (London, 1684), p. 275.

[11] Pierre Herigone, *Cursus mathematicus* (Paris, 1644), Vol. I, "Explicatio notarum" [1st ed., 1634].

[12] John Wallis, *Operum mathematicorum pars prima* (Oxford, 1657), p. 208.

[13] John Wallis, *op. cit.*, p. 334, 335.

[14] John Wallis, *Treatise of Algebra* (London, 1685), p. 127.

er." Probably Wallis' change in 1657 from Oughtred's $\backsim$ to $\sim$ was purely accidental. Jones[1] is faithful to Oughtred's $\backsim$ and uses also $\overset{+}{\backsim}$, but Barrow,[2] who usually adheres to Oughtred's symbols, adopts the symbol —: which he defines as "The Difference, or Excess; Also, that the Quantities which follow, are to be subtracted, the Signs not being changed."

In England and the United States the sign $\sim$ or $\backsim$ for "absolute difference" has been used down to the present time. Thus one finds $\backsim$ in texts by Clarke,[3] J. Thomson,[4] Bridge.[5] The sign $\sim$ is found in the writings of Morton[6] who, from $A:B::C:D$ derives $A\sim B:B::C\sim D:D$ and explains that $\sim$ means "the difference of the magnitudes which are represented by them, without supposing the first to be the greater, as is the case when we write $A-\ -B$ by itself." It is found also in Wright,[7] Goodwin,[8] Hall and Knight,[9] and C. Smith.[10]

The algebra of Alexander[11] uses the sign ⋎ for "absolute difference." W. P. G. Bartlett[12] (Cambridge, Mass.) stated in 1859, "I have used a German notation $\backsim$, to denote the difference between $\frac{y}{p}$ and $\frac{y}{y_0}$, because p may fall on the other side of y_0."

488. On the European continent the signs $\sim$ and $\backsim$ came to be employed for the expression of absolute difference by a few authors outside of Germany. In Germany these symbols were widely used for another purpose, namely, to express geometric similarity. Praalder[13] uses for arithmetic difference an ↼, looking like a letter *f* placed

[1] William Jones, *Synopsis palmariorum matheseos* (London, 1706), p. 248, 249.

[2] *Euclid's Elements .... Corrected by J. Barrow* (London, 1751), list of symbols.

[3] H. Clarke, *The Rationale of Circulation Numbers* (London, 1777), p. 207.

[4] James Thomson, *Treatise on Arithmetic* (18th ed.; Belfast, 1837), p. 101.

[5] B. Bridge, *The Three Conic Sections* (2d London ed.; New Haven, 1836), p. 9.

[6] [Pierce Morton], *Geometry, Plane Solid and Spherical* (London, 1830), p. 41.

[7] J. M. F. Wright, *Self-Examination in Euclid* (Cambridge, England, 1829).

[8] Harvey Goodwin, *Elementary Course in Mathematics, designed Principally for the Students of the University of Cambridge* (3d ed.; Cambridge 1869), p. 3.

[9] H. S. Hall and S. R. Knight, *Elementary Algebra* (ed. F. L. Sevenoak; New York, 1895), p. 374.

[10] C. Smith, *Elementary Algebra* (ed. Irving Stringham; New York, 1895), p. 18.

[11] *Synopsis algebraica, opus posthumum Johannis Alexandri, Bernatis-Helvetii* (London, 1693), p. 6.

[12] J. D. Runkle, *Mathematical Monthly*, Vol. I, No. 5 (Feb., 1859), p. 181.

[13] L. Praalder in his edition of the *Mathematische Voorstellen* of Ludolf van Keulen (Amsterdam, 1777), p. 205.

horizontally. The $\backsim$ for difference is found in Saverien,[1] Cagnoli,[2] and Da Cunha.[3] A Spaniard[4] writes $l\sim L$, $1\underset{\sim}{+}B$, $45°\underset{\sim}{+}F$.

The use of $=$ for absolute difference was noted in §§ 164, 177, 262.

489. *Other meanings of* $\backsim$ *and* $\sim$.—Argand[5] employed "$\sim a$" for our $a\sqrt{-1}$, and "$\backsim a$" for our $-a\sqrt{-1}$; Hudson and Lipka[6] use $\simeq$ for "equals approximately," which is in line with the older usage of $\sim$ for *annähernd gleich* by Günther,[7] M. Cantor, and others. With Peano[8] $\sim$ signifies "not"; $\sim\epsilon$, "is not"; and $\sim=$, "is not equal."

490. The sign $\sim$ is used extensively to express equivalence. G. Eisenstein[9] said in 1844, "Sind f and f' aequivalent, welches ich so bezeichne: $f\sim f'$." The sign is used in this sense by A. Wiegand.[10] The notation $a_\nu\sim b_\nu$ was used by P. du Bois-Reymond[11] in the case of $\lim\limits_{\nu=+\infty}\frac{a_\nu}{b_\nu}=a$, and by A. Pringsheim and J. Molk[12] when the series $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_\nu}{b_\nu}, \ldots$ diverges so as to have a superior limit and a distinct inferior limit, both finite and positive. André[13] attributes the use of $\sim$ for equivalence to Kronecker and his school. With G. H. Hardy and J. E.

[1] A. Saverien, *Dictionnaire universel de mathematique et de physique* (Paris 1753), art. "Caractere."

[2] Antonio Cagnoli, *Traité de Trigonométrie* ... Traduit de l'Italien par M. Chompré (Paris, 1786), p. 4.

[3] J. A. da Cunha, *Principios mathematicos* (Lisboa, 1790), p. 49.

[4] Dionisio Alcalá-Galciano, Capitan de Navío de la Real Armada, *Memoria sobre las observaciones de Latitud y Longitud en el Mar* (1796), p. 46.

[5] J. R. Argand, *Essai sur une manière de représenter les quantités imaginaires* (Paris, 1806). A. S. Hardy's translation (New York, 1881), p. 35.

[6] R. G. Hudson and J. Lipka, *A Manual of Mathematics* (New York, 1917), p. 68.

[7] S. Günther, "Die quadratischen Irrationalitäten der Alten," *Abhandlungen zur Geschichte der Mathematik*, Vol. IV (Leipzig, 1882), p. 11. Günther gives a reference to M. Cantor, "Gräko-indische Studien," *Zeitschr. f. Math. u. Phys., Hist. Lit. Abth.*, Band XXII, S. 1 ff.

[8] G. Peano, *Lezioni di analsisi infinitesimale*, Vol. I (Torino, 1893), p. 10.

[9] G. Eisenstein in *Crelle's Journal*, Vol. XXVII (1844), p. 91.

[10] August Wiegand, *Algebraische Analysis und Anfangsgründe der Differential-Rechnung* (2d ed.; Halle, 1853), p. 44.

[11] P. du Bois-Reymond in *Annali di matematica pura ed applicata*, serie II, Vol. IV (1870–71), p. 339.

[12] A. Pringsheim and J. Molk in *Encyclopédie des scien. math.*, Tom. I, Vol. I (1904), p. 201, 202.

[13] Désire André, *Notations mathématiques* (Paris, 1909), p. 98; L. Kronecker, *Vorlesungen über Zahlentheorie*, Vol. I (Leipzig, 1901), p. 86.

Littlewood[1] "$f(n) \sim g(n)$ means that $\frac{f}{g} \to 1$ when $n \to \infty$." Bromwich[2] in presenting Poincaré's theory of asymptotic series places $\sim$ between the function and the series under consideration, to denote that "the series is asymptotic to the function." He writes also $a_n \sim kb_n$ when $a_n/b_n \to k > 0$. Similarly with W. B. Ford[3], $f(x) \sim a_0 + a_1/x + a_2/x^2 + \cdots\cdot$ represents symbolically the relation $\lim\limits_{x=+\infty} x^n[f(x) - (a_0 + a_1/x + a_2/x^2 + \cdots\cdot + a_n/x^n] = 0$; $n = 0, 1, 2, 3, \ldots .$

491. *A few sporadic symbols.*—Kirkman[4] lets $(\pm)$ denote "that the signs have to be properly affixed, after the permutations are written out."

Simpson[5] uses the symbolism, "If $\underset{\shortparallel}{n}$, $\underset{\mid}{n}$, n, $\overset{\mid}{n}$, $\overset{\shortparallel}{n}$, $\overset{\mid\mid\mid}{n}$, &c. be a series of Terms in a decreasing Arithmetical Progression, whose common difference is $\dot{n}$. . . . ." Chauncey Lee[6] explains that ", = = =, the double line, drawn under a row of figures, shews that the operation is finished, and the answer stands over it."

492. *Signs for absolute value.*—There has been a real need in analysis for a convenient symbolism for "absolute value" of a given number, or "absolute number," and the two vertical bars introduced in 1841 by Weierstrass,[7] as in $|\, z\, |$, have met with wide adoption; it occurs in an article on the power series

$$F(x) = \sum_{\nu=-\infty}^{\nu=+\infty} A_\nu x^\nu ,$$

where x is a complex variable: "Ist dann r irgend eine bestimmte, innerhalb des Convergenzbezirks der Reihe liegende positive Grösse, so hat der absolute Betrag von $F(x)$, wenn man der Veränderlichen x

[1] G. H. Hardy and J. E. Littlewood, *Nachrichten von der K. Gesellschaft d. Wiss. zu Göttingen, Math.-Phys. Klasse* (1920), p. 34.

[2] T. J. I'A. Bromwich, *Introduction to the Theory of Infinite Series* (1908), p. 330. See also p. 11.

[3] W. B. Ford, *Studies on Divergent Series and Summability*, Michigan Science Series, Vol. II (New York, 1916), p. 22.

[4] T. P. Kirkman, *First Mnemonical Lessons in Geometry, Algebra and Trigonometry* (London, 1852), p. 166.

[5] Thomas Simpson, *Essays on . . . . Mathematics* (London, 1740), p. 87.

[6] Chauncey Lee, *The American Accomptant* (Lansingburgh, 1797), p. 64.

[7] K. Weierstrass, *Mathematische Werke*, Vol. I (Berlin, 1894), p. 67.

alle diejenigen Werthe beilegt, für welche $|x|=r$ ist, eine obere Grenze, die mit g bezeichnet werde: und es gilt der Satz: $|A_\mu| \leqq gr^{-\mu}$ für jeden ganzzahligen Werth von μ." This article was not printed at the time. Weierstrass used $|\ |$ again in an article read before the Berlin Academy of Sciences on December 12, 1859.[1] Weierstrass also employed the notation $|\ |$ for determinants,[2] and avoided confusion by the use of such phrases as "Die Determinante $|\omega_{a_y}|$."

André[3] states that the mark $\bar{z}$ has also been used to designate absolute value. The abbreviation[4] "mod z" from Argand's and Cauchy's word[5] "module" is not very convenient; nor is it desirable in view of the fact that "mod" has been widely adopted, since the time of Gauss, in the theory of numbers.

493. *Zeroes of different origin.*—Riabouchinski[6] gives a new extension of the notion of number leading to transfinite numbers differing from those of G. Cantor; he represents the ordinary symbolism $\lim\limits_{n\to\infty}\frac{a}{n}=\text{o}$ by the symbol $\frac{a}{J}=\pm\text{o}$. As a is taken arbitrarily, he assumes the existence of an infinity of zeroes of different origin; if a is positive, the corresponding zero is marked $+\text{o}$ or o, if negative the zero is marked $-\text{o}$. Let $\frac{1}{J}$ be $\dot{\text{o}}$, and J the inverse of passage to the limit, then $J\cdot\frac{5}{J}=J\cdot\text{o}=5$. When the origin of a zero is not defined, he writes $J\cdot\pm\text{o}=a$, where a is arbitrary. He has $a\cdot\text{o}=\pm\text{o}$, $J=\frac{1}{\dot{\text{o}}}$, $J\cdot a=\frac{a}{\dot{\text{o}}}$, $a\cdot\text{o}=\frac{ab}{J}=ab\cdot\dot{\text{o}}=\pm\text{o}$, $1+J=J+1$. He designates by $\pm\overline{|\quad}$ the operation of the return to relative values or the inverse of the passage to absolute values. Accordingly, he lets $\overline{|-1}$ or j represent the result of an impossible return to a relative value. Then, $|j|=-1$, $j=\pm\overline{|-1}$.

[1] K. Weierstrass, *op. cit.*, Vol. I, p. 252.

[2] K. Weierstrass, *op. cit.*, Vol. IV, p. 524.

[3] D. André, *op. cit.*, p. 86.

[4] See G. Peano, *Formulaire mathématique*, Vol. IV (1903), p. 93.

[5] A. L. Cauchy, *Exercices de mathématiques* (Paris, 1829), Tome IV, p. 47; *Œuvres* (2d ser.), Vol. IX, p. 95.

[6] D. Riabouchinski, "Sur le calcul des valeurs absolues," *Comptes rendus du congrès international des mathématiciens* (Strasbourg, 22–30 Septembre 1920; Toulouse, 1921), p. 231–42.

494. *General combinations between magnitudes or numbers.*—In the establishment of a combination (*Verknüpfung*) between magnitudes or numbers, symbols like $a \frown b$ and $a \smile b$ were used by H. G. Grassmann[1] in 1844, where $\frown$ indicated an affirmative proposition or thesis, and $\smile$ a corresponding negative proposition or lysis. A second thesis was marked $\overset{\frown}{\frown}$. The two symbols for thesis and lysis were easily confused, one with the other, so that his brother R. Grassmann[2] used a small circle in place of the $\frown$; he used also a large circle with a dot at its center $\odot$, to mark a second relationship, as in $(a \circ e)b = ab \odot ae$. Likewise Stolz[3] used the small circle for the thesis and H. G. Grassmann's $\smile$ for the lysis. Hankel[4] in 1867 employed for thesis and lysis the rather cumbrous signs $\Theta\ (a, b)$ and $\lambda(a, b)$. R. Bettazzi[5] writes $S(a, b)$ and the inverse $D(a, b)$. The symbol $a \circ b = c$ is used by Stolz and Gmeiner[6] to designate "a mit b ist c"; the small circle indicates the bringing of a and b into any relationship explained by c; the $=$ expresses here "is explained by." When two theses[7] are considered, the second is marked $\odot$; a second lysis is marked $\dot{\smile}$. When $x \circ b = a$, $b \circ y = a$ are both single valued, they are marked respectively $x = a \smile b$, $y = a \underline{\smile} b$. The small circle is used also by Huntington.[8] He says: "A system composed of a class K and a rule of combination $\circ$ we shall speak of as a system $(K, \circ)$." When different classes come under consideration, different letters K, C represent them, respectively. When different rules of combination are to be indicated, a dot or small $+$ or $<$ is placed within the circle. Thus Huntington speaks of the "system $(K, C, \oplus, \odot, \bigcirc\!\!\!\!<\,)$." Huntington also writes $A^{[n]}$ to represent $A \circ A \circ \ldots . \circ A$, where n is a positive element. Dickson[9] marks by $\oplus$ the operation of addition, by $\odot$ that of multiplication, and by $\bigcirc$ the special operation of scalar multiplication.

[1] H. G. Grassmann, *Die lineale Ausdehnungslehre* (1844; 2d ed., 1878), §§ 5 f.

[2] Robert Grassmann, *Die Grössenlehre* (Stettin, 1872), p. 27, 37.

[3] Otto Stolz, *Allgemeine Arithmetik* (Leipzig, 1885), p. 26, 28.

[4] H. Hankel, *Theorie der complexen Zahlensysteme* (Leipzig, 1867), p. 18–34.

[5] R. Bettazzi, *Theoria delle grandezze* (Pisa, 1890), p. 7, 11.

[6] O. Stolz und J. A. Gmeiner, *Theoretische Arithmetik* (Leipzig), Vol. I (2d ed., 1911), p. 7, 51–58.

[7] O. Stolz und J. A. Gmeiner, *op. cit.*, p. 63, 66.

[8] J. W. A. Young's *Monographs* (New York, 1911), p. 164, 186–99. See also E. V. Huntington in *Trans. Amer. Math. Soc.*, Vol. VI (1905), p. 209–29.

[9] L. E. Dickson, *Algebras and Their Arithmetics* (Chicago, 1923), p. 9.

SYMBOLISMS FOR IMAGINARIES AND VECTOR ANALYSIS

495. *Symbols for the square root of minus one.*—Nicolas Chuquet in his manuscript work of 1484, *Le Triparty,* solved $4+x^2=3x$ and obtained $x=\frac{3}{2}\pm\sqrt{2\frac{1}{4}-4}$ (expressed here in modern symbols). As $2\frac{1}{4}-4$ is negative, he pronounced the root impossible. In his own words: "Et pourtant que .2.$\frac{1}{4}$. qui est la multiplicacion du moyen est moindre que le precedent Il senβ que ceste raiβ est impossible."[1] Chuquet did not admit imaginary roots as valid, and in the solution of the equation quoted, does not use symbolism for square root. In general, he indicates square root as in "R^2.21" for $\sqrt{21}$.

The earliest mathematician seriously to consider imaginaries and to introduce them in the expression for the roots of equations was H. Cardan. He uses[2] the designation "R.$\tilde{m}$.," that is, *radix minus,* to express the square root of a negative number. The marking of the imaginary roots in the collected works of 1663 is not quite the same as in the *Ars magna* of 1545. In 1545[3] he gives the following multiplication:

$$\begin{array}{lcl} 5p{:}R\!\!\!/ m{:}15 & \qquad & 5+\sqrt{-15} \\ 5m{:}R\!\!\!/ m{:}15 & & 5-\sqrt{-15} \\ \hline 25m{:}m{:}15\ \tilde{q}d\ est\ 40 & & 25-(-15)=40 \end{array}$$

In 1663 this passage is given thus:

$$\begin{array}{l} 5.\tilde{p}\ .R\!\!\!/.\tilde{m}.15. \\ 5.\tilde{m}.R\!\!\!/.m.15. \\ \hline 25m.m.15.quad.est\ 40. \end{array}$$

It must be noted that Cardan doubted the validity of arithmetical operations performed upon imaginary quantities.

496. Apparently it was Rafaele Bombelli who undertook to "demonstrate as valid in every case the formula of Scipio del Ferro" ("dimostrare valida in ogni caso la formula di Scipione Dal Ferro").[4] Bombelli's algebra was issued in manuscript edition[5] about 1550 and

[1] *Bullettino Boncompagni,* Vol. XIII (1880), p. 805.

[2] H. Cardan, *Ars magna* (Nuremberg, 1545), chap. xxxvii; (nouv. éd.; Bâle, 1570), p. 130; *Opera* (Lyon, 1663), p. 287.

[3] J. Tropfke, *op. cit.,* Vol. II (2d ed., 1921), p. 18; Vol. III (2d ed., 1922), p. 135.

[4] See E. Bortolotti in *Scientia,* Vol. XXXIII (1923), p. 389.

[5] A copy of the manuscript edition has only recently been found by E. Bortolotti in the *Biblioteca Comunale of Bologna* (Codez B, 1569).

in printed form in 1572 and 1579. In the manuscript edition Bombelli writes the equation $x^2+20=8x$ in the form $|\overset{2}{\smile}p.20\ \grave{a}\ 8\overset{1}{\smile}$, and writes the root $4+\sqrt{-4}$ in the form $4.p.R[\underline{0\tilde{m}.4}]$ and the root $4-\sqrt{-4}$ in the form $4.\tilde{m}R[\underline{0\tilde{m}.4}]$. Similarly, he writes the cubic $x^3=15x+4$ in the form $|\overset{3}{\smile}.\grave{a}\ 15\overset{1}{\smile}.p.4\overset{0}{\smile}$ and its roots $R^3[\underline{2.p.R[\underline{0\tilde{m}\ 121}]}].p.R^3[\underline{2\tilde{m}R[\underline{0\tilde{m}.121}]}]$.

Further on, in his manuscript edition, Bombelli introduces a new phraseology and a new symbolism for the imaginary.[1] He says: "As one cannot call it either plus, or minus, I shall call it 'più di meno' when it is added, and 'meno di meno' when it is subtracted." This "più de meno" appears as the contraction for *più radice di meno;* the "meno di meno" for *meno radice di meno.* The "di meno" was abbreviated *d.m.* and stands for our $\sqrt{-1}$. Bombelli wrote "*p.dm.*2·*via.p.dm.* 1 *egual.*$\tilde{m}$ 2," which means $(+2i)(+i)=-2$; he wrote "1 *diviso da p. dm.* 1 *egual. m. dm.* 1," which means $1\div i=-i$.

That Bombelli was far in advance of his time on the algorism of imaginaries is strikingly evident, when one recalls that as late as the eighteenth century gross errors were committed. Even in Euler's *Elements of Algebra*[2] one finds $\sqrt{-2}.\sqrt{-3}=\sqrt{6}$, $\sqrt{-1}.\sqrt{-4}=2$, $\sqrt{+3}\div\sqrt{-3}=\sqrt{-1}$, $1\div\sqrt{-1}=\sqrt{-1}$. Euler was blind when he prepared this book, and he is probably not responsible for printers' errors.

497. Albert Girard in 1629 found the roots of the equation $x^4=4x-3$, using the symbolism $\sqrt{-2}$, in the following passage: "Si 1 (4) est esgale à 4 (1) -3, alors les quatre factions seront 0, 0, 4, 3, partant les quatre solutions seront 1, $1, -1+\sqrt{-2}, -1-\sqrt{-2}$."[3]

He speaks of three kinds of roots of equations, namely, those "qui sont plus que rien; d'autres moins que rien; & d'autres envelopées, comme celles qui ont des $\sqrt{-}$, comme des $\sqrt{-3}$, ou autres nombres semblables." In Wallis[4] one finds the signs $\sqrt{-3}$, $\sqrt{-e}$, also in Isaac Newton[5] $\sqrt{-2}$. Before Euler the sign $\sqrt{-1}$, as distinct from $\sqrt{-a}$, seldom, if ever, occurs.

[1] See E. Bortolotti, *op. cit.*, p. 391, 394.

[2] Leonhard Euler, *Vollständige Anleitung zur Algebra*, Erster Theil (St. Petersburg, 1770), §§ 148, 149, p. 87, 88.

[3] Albert Girard, *Invention nouvelle en l'Algebre* (Amsterdam, 1629), fol. *F A* and *F B*.

[4] John Wallis, *Treatise of Algebra* (1685), p. 180.

[5] Sir Isaac Newton, *Universal Arithmetick* (London, 1728), p. 193.

Kästner[1] was among the first to represent a pure imaginary by a letter; for $\sqrt[a]{-1}$ he wrote π where a is an even integer. He speaks of "den unmöglichen Factor $b+f\sqrt[a]{-1}$ für den ich $b+f\pi$ schreiben will." Before this he had used π for $\sqrt{a}$, where a was positive.[2]

Caspar Wessel[3] in an *Essay* presented to the Danish Academy in 1797 designates "by $+1$ positive rectilinear unity, by $+\epsilon$ another unity, perpendicular to the first and having the same origin," and then writes "$\sqrt{-1}=\epsilon$"; "$\cos v+\epsilon \sin v$."

498. It was Euler who first used the letter i for $\sqrt{-1}$. He gave it in a memoir presented in 1777 to the Academy at St. Petersburg, and entitled "De formulis differentialibus etc.," but it was not published until 1794 after the death of Euler.[4]

As far as is now known, the symbol i for $\sqrt{-1}$ did not again appear in print for seven years, until 1801. In that year Gauss[5] began to make systematic use of it; the example of Gauss was followed in 1808 by Kramp.[6]

499. Argand,[7] in his *Essai* (Paris, 1806), proposes the suppression of $\sqrt{-1}$ and the substitution for it of particular signs similar to the signs of operation $+$ and $-$. He would write $\sim$ for $+\sqrt{-1}$ and $\backsim$ for $-\sqrt{-1}$, both indicating a rotation through 90°, the former

[1] A. G. Kästner, *Anfangsgründe der Analysis endlicher Grössen* (Göttingen, 1760), p. 133.

[2] A. G. Kästner, *op. cit.*, p. 117.

[3] Caspar Wessel, *Essai sur la représentation analytique de la direction* (Copenhague, 1897), p. 9, 10, 12. This edition is a translation from the Danish; the original publication appeared in 1799 in Vol. V of the *Nye Samling af det Kongelige Danske Videnskabernes Selskabs Skrifter*.

[4] The article was published in his *Institutiones calculi integralis* (2d ed.), Vol. IV (St. Petersburg, 1794), p. 184. See W. W. Beman in *Bull. Amer. Math. Soc.*, Vol. IV (1897–98), p. 274. See also *Encyclopédie des scien. math.*, Tom. I, Vol. I (1904), p. 343, n. 60.

[5] K. F. Gauss, *Disquisitiones arithmeticae* (Leipzig, 1801), No. 337; translated by A. Ch. M. Poullet-Delisle, *Recherches arithmétiques* (Paris, 1807); Gauss, *Werke*, Vol. I (Göttingen, 1870), p. 414.

[6] C. Kramp, *Élémens d'arithmétique universelle* (Cologne, 1808), "Notations."

[7] J. R. Argand, *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (Paris, 1806; published without the author's name); a second edition was brought out by G. J. Hoüel (Paris, 1874); an English translation by A. S. Hardy appeared in New York in 1881. We quote from Hardy's edition, p. 35, 45.

in the positive direction, the latter in the negative. De Moivre's formula appears in Argand's *Essai* in the garb

$$\cos na \sim \sin na = (\cos a \sim \sin a)^n .$$

In 1813 Argand[1] interpreted $\sqrt{-1}^{\sqrt{-1}}$ as a sign of a unit vector perpendicular to the two vectors 1 and $\sqrt{-1}$. J. F. Français at first entertained the same view, but later rejected it as unsound.

W. R. Hamilton[2] in 1837 states that one may write a "couple" (a_1, a_2) thus: $(a_1, a_2) = a_1 + a_2\sqrt{-1}$. Accordingly, (0, 1) represents $\sqrt{-1}$. Similarly, Burkhardt[3] wrote the *Zahlenpaar* (a, b) for $a+bi$.

In manuscripts of J. Bolyai[4] are used the signs $+$, $-$, $\dotplus$, $\dot{-}$ for $+$, $-$, $\sqrt{-1}$, $-\sqrt{-1}$, but without geometrical interpretation.

Français[5] and Cauchy[6] adopted a_α as the notation for a vector. According to this, the notation for $\sqrt{-1}$ would be $1_{\frac{\pi}{2}}$. However, in 1847 Cauchy began to use i in a memoir on a new theory of imaginaries[7] where he speaks of " ... le signe symbolique $\sqrt{-1}$, auquel les géomètres allemands substituent la lettre i," and continues: "Mais il est évident que la théorie des imaginaires deviendrait beaucoup plus claire encore et beaucoup facile à saisir, qu'elle pourrait être mise à la portée de toutes les intelligences, si l'on parvenait à réduire les expressions imaginaires, et la lettre i elle-même, a n'être plus que des quantités réelles." Using in this memoir the letter i as a *lettre symbolique,* substituted for the variable x in the formulas considered, he obtains an *équation imaginaire,* in which the symbolic letter i should be considered as a real quantity, but indeterminate. The symbolic letter i, substituted for x in an integral function $f(x)$, indicates the value received, not by $f(x)$, but by the rest resulting from the alge-

[1] J. R. Argand, *Annales Math. pures appl.*, Vol. IV (1813–14), p. 146. See also *Encyclopédie des scien. math.*, Tom. I, Vol. I (1904), p. 358, n. 88.

[2] W. R. Hamilton, *Trans. Irish Acad.*, Vol. XVII (1837), p. 417, 418; *Lectures on Quaternions* (Dublin, 1853), Preface, p. 10.

[3] H. Burkhardt, *Einführung in die Theorie der analytischen Funktionen* (Leipzig, 1903), p. 3.

[4] See P. Stäckel, *Math. Naturw. Ber. Ungarn*, Vol. XVI (1899), p. 281. See also *Encyclopédie des scien. math.*, Tom. I, Vol. I (1904), p. 346, n. 63.

[5] J. F. Français in *Annalen Math. pures appl.*, Vol. IV (1813–14), p. 61.

[6] A. L. Cauchy in *Comptes rendus Acad. sci.* (Paris), Vol. XXIX (1849), p. 251; *Œuvres* (1st ser.), Vol. XI (Paris, 1899), p. 153.

[7] A. L. Cauchy, *Comptes rendus*, Vol. XXIV (1847), p. 1120 = *Œuvres complètes* (1st ser.), Vol. X, p. 313, 317, 318.

braic division of $f(x)$ by x^2+1. He obtains in a particular case $i^2+1=0$, and equations like $(a+bx)(c+dx)=ac+bdx^2+(ad+bc)x$ which gives, when one writes i for x $(a+bi)(c+di)=ac-bd+(ad+bc)i$. Cauchy[1] used the letter i for $\sqrt{-1}$ in papers of 1851 and later.

That the symbol i for $\sqrt{-1}$ was somewhat slow in securing general adoption is illustrated by the fact that as alert a writer as A. de Morgan used k for $\sqrt{-1}$ in his *Differential and Integral Calculus* of 1842 (p. 118). J. M. Peirce's form is shown in Figure 108 of § 401. A. C. Dixon[2] writes ι for $\sqrt{-1}$.

500. With the development of electromagnetism physicists and electricians, C. P. Steinmetz[3] for instance, let the letter $\underset{\cdot}{i}$ stand for strength of an electric current. In presenting the mathematical theory of electromagnetism they avoided confusion resulting from the double use of i by letting j represent $\sqrt{-1}$. For example, in a recent *Manual*,[4] one finds the notation $\underset{\cdot}{x}=a+jb$, where $j=\sqrt{-1}$ and $\underset{\cdot}{x}$ is a vector. On the other hand, Bryan[5] introduces j for $\log-1$, in aeroplane calculations of the products of quantities, some positive and some negative, by the use of logarithmic tables; he writes $\log-2=j0{\cdot}30103$, $\log-0{\cdot}02=j\bar{2}{\cdot}30103$. An even number of j's yields a positive product. For $\sqrt{-1}$ Bryan uses i or ι.

501. *De Morgan's comments on* $\sqrt{-1}$.—The great rôle that $\sqrt{-1}$ has played in the evolution of algebraic theory is evident from the statements of Augustus de Morgan. In 1849 he spoke of "the introduction of the unexplained symbol $\sqrt{-1}$,"[6] and goes on to say: "The use, which ought to have been called *experimental*, of the *symbol* $\sqrt{-1}$, under the name of an *impossible* quantity, shewed that; come how it might, the intelligible results (when such things occurred) of the experiment were always true, and otherwise demonstrable. I am now going to try some new experiments." Consider also the following passage from De Morgan: "As soon as the idea of acquiring symbols

[1] See E. B. Jourdain, *Bibliotheca mathematica* (3d ser., Vol. VI (1905–6), p. 191, 192 n.; A. Brill and M. Noether, "Entwicklung der Theorie der algebr. Funktionen," *Jahresbericht d. d. Mathematiker Vereinigung*, Vol. III (1894), p. 194.

[2] A. C. Dixon, *Elliptic Functions* (London, 1894), p. 5.

[3] C. P. Steinmetz, *Alternating Current Phenomena* (New York, 1908), p. 34 ff.

[4] R. G. Hudson and J. Lipka, *A Manual of Mathematics* (New York, 1917), p. 68.

[5] G. H. Bryan, *Mathematical Gazette*, Vol. VIII (1917), p. 220.

[6] Augustus de Morgan, *Trigonometry and Double Algebra* (London, 1849), p. 41.

and laws of combination, without giving meaning, has become familiar, the student has the notion of what I will call a *symbolic calculus;* which, with certain symbols and certain laws of combination, is *symbolic algebra:* an art, not a science; and an apparently useless art, except as it may afterwards furnish the grammar of a science. The proficient in a symbolic calculus would naturally demand a supply of meaning."[1] And again: "The first who used algebraical symbols in a general sense, Vieta, concluded that subtraction was a defect, and that expressions containing it should be in every possible manner avoided. *Vitium negationis,* was his phrase. Nothing could make a more easy pillow for the mind, than the rejection of all which could give any trouble; .... The next and second step, ... consisted in treating the results of algebra as necessarily true, and as representing some relation or other, however inconsistent they might be with the suppositions from which they were deduced. So soon as it was shewn that a particular result had no existence as a quantity, it was permitted, by definition, to have an existence of another kind, into which no particular inquiry was made, because the rules under which it was found that the new symbols would give true results, did not differ from those previously applied to the old ones. .... When the interpretation of the abstract negative quantity shewed that a part at least of the difficulty admitted a rational solution, the remaining part, namely that of the square root of a negative quantity, was received, and its results admitted, with increased confidence."[2]

502. *Notation for vector.*—Argand[3] adopted for a directed line or vector the notation $\overline{AB}$. Möbius[4] used the designation AB, where A is the origin and B the terminal point of the vector. This notation has met with wide adoption. It was used in Italy by G. Bellavitis,[5] in France by J. Tannery,[6] by E. Rouché and Ch. de Comberousse,[7]

[1] A. de Morgan, *op. cit.*, p. 92.

[2] A. de Morgan, *op. cit.*, p. 98, 99.

[3] J. R. Argand, *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (Paris, 1806).

[4] A. F. Möbius, *Der barycentrische Calcul* (Leipzig, 1827); *Werke,* Vol. I (Leipzig, 1885), p. 25.

[5] G. Bellavitis, "Metodo delle Equipollenze," *Annali delle scienze del regno Lombardo-Veneto* (Padova), Vol. VII (1837), p. 243–61.

[6] J. Tannery, "Deux leçons de cinématique," *Ann. École Normal* (3d ser.), Vol. III (1886), p. 43–80.

[7] E. Rouché et Ch. de Comberousse, *Traité de géométrie* (7th ed.), Vol. I (Paris, 1900), p. 216.

and P. Appell.[1] On the other hand, W. R. Hamilton, P. G. Tait, and J. W. Gibbs designated vectors by small Greek letters. D. F. Gregory used the symbol $a(.)$ to "represent a straight line, as the result of transferring a point through a given space in a constant direction."[2] We have already called attention to the designation a_a of Français and Cauchy, which is found in some elementary books of the present century; Frank Castle[3] lets $A\alpha$ designate a vector. Frequently an arrow is used in marking a vector, thus $\overrightarrow{R}$, where R represents the length of the vector. W. Voigt[4] in 1896 distinguished between "polar vectors" and "axial vectors" (representing an axis of rotation). P. Langevin[5] distinguishes between them by writing polar vectors $\overrightarrow{a}$, axial vectors $\overset{\circlearrowright}{a}$. Some authors, in Europe and America, represent a vector by bold-faced type $\boldsymbol{a}$ or $\boldsymbol{b}$ which is satisfactory for print, but inconvenient for writing.

Cauchy[6] in 1853 wrote a radius vector $\bar{r}$ and its projections upon the co-ordinate axes $\bar{x}, \bar{y}, \bar{z}$, so that $\bar{r}=\bar{x}+\bar{y}+\bar{z}$. Schell[7] in 1879 wrote $[AB]$; some others preferred (AB).

503. Wessel[8] in 1797 represented a vector as a sum $x+\eta y+ez$, with the condition that $\eta^2=-1$, $e^2=-1$. H. G. Grassmann[9] represents a point having x, y, z for its co-ordinates by $p=v_1e_1+v_2e_2+v_3e_3$, where the e_1, e_2, e_3 are unit lengths along the co-ordinate axes. W. R. Hamilton[10] wrote $\rho=ix+jy+kz$, where i, j, k are unit vectors per-

[1] P. Appell, *Traité de mécanique rationnelle* (3d ed.), Vol. I (Paris, 1909), p. 3.

[2] D. F. Gregory, *Cambridge Math. Jour.*, Vol. II (1842), p. 1; *Mathematical Writings* (Cambridge, 1865), p. 152.

[3] Frank Castle, *Practical Mathematics for Beginners* (London, 1905), p. 289. We take these references from *Encyclopédie des scien. math.*, Tom. IV, Vol. II (1912), p. 4.

[4] W. Voigt, *Compendium der theoretischen Physik*, Vol. II (Leipzig, 1896), p. 418–801.

[5] *Encyclopédie des scien. math.*, Tom. IV, Vol. II (1912), p. 25.

[6] A. L. Cauchy, *Comptes rendus Acad. sc.*, Vol. XXXVI (Paris, 1853), p. 75; *Œuvres* (1st ser.), Vol. XI (Paris, 1899), p. 443.

[7] W. Schell, *Theorie der Bewegung der Kräfte* (2d ed.), Vol. I (Leipzig, 1879).

[8] C. Wessel, "Om directionens analytiske Betegning," *Nye Samling af det Kongelige Danske Videns-Kabernes Selskabs Skrifter* (2), Vol. V (1799), p. 469–518. French translation (Copenhagen, 1897), p. 26. See *Encyclopédie des scien. math.*, Tom. IV, Vol. II (1912), p. 12.

[9] H. G. Grassmann, *Ausdehnungslehre von 1844* (2d ed.; Leipzig, 1878), p. 128, § 92.

[10] W. R. Hamilton, *Lectures on Quaternions* (Dublin, 1853), p. 59 (§ 65), p. 105 (§ 101).

pendicular to one another, such that $i.j=-j.i=k$, $j.k=-k.j=i$, $k.i=-ik=j$.

Stringham[1] denoted $\cos\beta+i\sin\beta$ by "cis β," a notation used also by Harkness and Morley.[2] Study[3] represented vectorial quantities by a biplane, or two non-perpendicular planes, the initial plane ϕ, and the final plane ϕ'. The biplane is represented by $\mathfrak{K}_{\phi}^{\phi'}$. Similarly, Study defines a "motor" by two non-perpendicular straight lines ($\mathfrak{X}$ the initial line, and $\mathfrak{Y}$ the final), and represents the motor by the symbol $\mathfrak{M}_{\mathfrak{X}}^{\mathfrak{Y}}$.

504. *Length of vector.*—The length of a vector was marked by Bellavitis AB (the same designation as for vector), by H. G. Grassmann[4] $\sqrt{R^2}$, by W. R. Hamilton[5] TR (i.e., tensor of the vector R), by R. Gans[6] $|R|$, the Weierstrassian symbol for absolute value. Later Gans discarded $|R|$ because $|R|$ is a function of R and functional symbols should precede or follow the entity affected, but $|R|$ does both.

505. *Equality of vectors.*—L. N. M. Carnot[7] employed $\doteqdot$ as *signe d'équipollence*, practically a sign of identity. "Si les droites $\overline{AB}$, $\overline{CD}$ concurrent au point E, j'écrirai $\overline{AB}\cdot\overline{CD}\doteqdot E$," where $\overline{AB}\cdot\overline{CD}$ means the point of intersection of the two lines.

To express the equality of vectors the sign $=$ has been used extensively. Möbius[8] had a few followers in employing $\equiv$; that symbol was used by H. G. Grassmann[9] in 1851. Bellavitis[10] adopted the astronomical sign *libra* $\simeq$, H. G. Grassmann[11] in 1844 wrote $\#$, which was also chosen by Voigt[12] for the expression of complete equality. Hamil-

[1] Irving Stringham, *Uniplanar Algebra* (San Francisco, 1893), p. xiii, p. 101.

[2] J. Harkness and F. Morley, *Theory of Analytic Functions* (London, 1898), p. 18, 22, 48, 52, 170.

[3] E. Study, *Geometrie der Dynamen* (Leipzig, 1903), p. 30, 51; *Encyclopédie des scien. math.*, Tom. IV, Vol. II (1912), p. 55, 59.

[4] H. Grassmann, *Werke*, Vol. I[1] (Leipzig, 1894), p. 345; Vol. 1[2] (1896), p. 118.

[5] W. R. Hamilton, *Cambr. and Dublin Math. Jour.*, Vol. I (1846), p. 2.

[6] R. Gans, *Einführung in die Vector Analysis* (Leipzig, 1905), p. 5.

[7] L. N. M. Carnot, *Géométrie de position* (Paris, 1803), p. 83, 84.

[8] A. F. Möbius, *Der baryc. Calcul* (1827), § 15; *Werke*, Vol. I, p. 39.

[9] *Crelle's Journal*, Vol. XLII, p. 193–203; Hermann Grassmann's *Gesammelte math. und physik. Werke*, Band II (ed. F. Engel; Leipzig, 1904), p. 90.

[10] G. Bellavitis, *op. cit.*, p. 243–61.

[11] Hermann Grassmann, *Gesammelte . . . . Werke*, Band I; *Ausdehnungslehre* (von 1844), p. 67.

[12] W. Voigt, *Nachricht. K. Gesellsch. d. Wissensch. z. Göttingen* (1904), p. 495–513.

ton[1] in 1845 wrote, ". . . . the symbolic equation, $D-C=B-A$, may denote that the point D is ordinarily related (in space) to the point C as B is to A, and may in that view be also expressed by writing the *ordinal analogy*, $D..C::B..A$; which admits of *inversion* and *alternation*." Peano[2] employed the symbolism $a-b=c-d$.

506. *Products of vectors.*—Proceeding to the different products of vectors, one observes that there has been and is great variety of notations. W. R. Hamilton assigned to two vectors only one product $\rho\rho'$, which is a quaternion and is the sum of two parts, the scalar of the quaternion $S\rho\rho'$, and the vector part of the quaternion $V\rho\rho'$.

H. G. Grassmann developed several products, of which the "internal product" or scalar product (the same as Hamilton's $S\rho\rho'$) and the "external product" or vector product (the same as Hamilton's $V\rho\rho'$) occupy a central place in vector analysis. Grassmann[3] represented in 1846 the scalar product by $a\times b$; in 1862[4] the scalar product by $[u|v]$ and in 1844 and 1862 the vector product by $[uv]$.

Résal[5] wrote for the scalar product $a\times b$, as did also Peano[6] in 1899, and later Burali-Forti and Marcolongo.[7] At an earlier date, 1888, Peano had written the scalar product $u|v$. Burali-Forti and Marcolongo give the vector product in the form $u\wedge v$. The simple form uv for the scalar product is employed by Somov,[8] Heaviside,[9] Föppl,[10] Ferraris.[11] Heaviside at one time followed Hamilton in writing the vector product, Vuv. Gibbs[12] represented the scalar product, called also "dot product," by $u.v$, and the vector product by $u\times v$; Gibbs's $u.v=-Suv$ of Hamilton.

[1] W. R. Hamilton, *Cambr. and Dublin Math. Jour.*, Vol. I (1846), p. 47.

[2] G. Peano, *Formulaire mathématique*, Vol. IV (1903), p. 253, 254.

[3] H. G. Grassmann, *Geometrische Analyse* (Leipzig, 1847); *Werke*, Vol. I[1] (Leipzig, 1894), p. 345.

[4] H. Grassmann, *Werke*, Vol I[1], p. 345; Vol. I[2], p. 56, 112.

[5] H. Résal, *Traité de cinématique pure* (Paris, 1862).

[6] G. Peano, *Formulaire de mathématiques*, Vol. II (Turin, 1899), p. 156.

[7] C. Burali-Forti et R. Marcolongo, *Elementi di calcolo vettoriale* (Bologna, 1909), p. 31. See *Encyclopédie des scien. math.*, Tom. IV, Vol. II, p. 14, 22.

[8] P. Somov, *Vector Analysis and Its Applications* (Russian) (St. Petersburg, 1907).

[9] O. Heaviside, *Electrical Papers*, Vol. II (London, 1892), p. 5.

[10] Föppl, *Geometrie der Wirbelfelder* (1897).

[11] G. Ferraris, *Lezioni di elettratecnica*, Kap. I (1899).

[12] J. W. Gibbs's *Vector Analysis*, by E. B. Wilson (New York, 1902), p. 50.

H. A. Lorentz[1] writes the scalar product $(u.v)$ and the vector product $[u.v]$; Henrici and Turner[2] adopted for the scalar product (u,v); Heun[3] has for the scalar product $\bar{u}\,\bar{v}$, and $\overline{uv}$ for the vector product; Timerding and Lévy, in the *Encyclopédie*,[4] use $u|v$ for the scalar product, and $u\times v$ for the vector product. Macfarlane at one time suggested cos ab and sin ab to mark scalar and vector products, to which Heaviside objected, because these products are more primitive and simpler than the trigonometric functions. Gibbs considered also a third product,[5] the *dyad*, which was a more extended definition and which he wrote uv without dot or cross.

507. *Certain operators.*—W. R. Hamilton[6] introduced the operator $\nabla = i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}$, a symbolic operator which was greatly developed in the later editions of P. G. Tait's *Treatise on Quaternions.* The sign ∇ was called "nabla" by Heaviside, "atled" by others, which is "delta," Δ, reversed. It is often read "del." The vector $-\nabla\phi$ is sometimes called the "gradient"[7] of the scalar ϕ and is represented by Maxwell and Riemann-Weber, as grad ϕ. The form $-S\nabla a$ is the quaternion representation of what Abraham and Langevin call div $\vec{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}$, and what Gibbs marked $\nabla . a$, and Heaviside ∇a. The mark $V\nabla a$, used by Tait, is Gibbs's $\nabla\times a$, Heaviside's "curl a," Wiechert's "Quirl a," Lorentz' "Rot a," Voigt's "Vort a," Abraham and Langevin's[8] "Rot $\vec{a}$"; Clifford's "divergence" of u, marked *div* u or *dv* u, is the $-S\nabla u$ of Hamilton[9] and the $\nabla . x$ of Gibbs.[10]

[1] See *Encyclopédie des scien. math.*, Tom. IV, Vol. II, p. 14, 22.

[2] O. Henrici and G. C. Turner, *Vectors and Rotors* (London, 1903).

[3] K. Heun, *Lehrbuch der Mechanik,* Vol. I (Leipzig, 1906), p. 14.

[4] *Encyclopédie des scien. math.*, Tom. IV, Vol. II (1912), p. 14, 22.

[5] See the Gibbs-Wilson *Vector-Analysis* (1901), chap. v.

[6] W. R. Hamilton, *Lectures on Quaternions* (1853), p. 610. The rounded letter ∂, to indicate that the derivatives in this operator are partial, is found in A. McAulay's *Octonions,* Cambridge, 1898, p. 85, and in C. J. Joly's *Quaternions,* London, 1905, p. 75.

[7] See, for instance, the *Encyclopédie des scien. math.*, Tom. IV, Vol. V, p. 15, 17, 18. Fuller bibliographical references are given here. C. Burali-Forti in *L'Enseignement mathématique,* Vol. XI (1909), p. 41.

[8] See, for instance, the *Encyclopédie des scien. math.*, Tom IV, Vol. V, p. 15, 17, 18, where fuller bibliographical references are given.

[9] C. Burali-Forti et R. Marcolongo, *op. cit.*, Vol. XI, p. 44.

[10] For additional historical details about ∇ see J. A. Schouten, *Grundlagen der Vector- und Affinoranalysis* (1914), p. 204–13.

508. *Rival vector systems.*—W. R. Hamilton and H. G. Grassmann evolved their algebras about the same time (1843, 1844). Hamilton's quaternions received the attention of Tait and some other mathematicians in Great Britain; there it was applied to physical problems. Grassmann's *Ausdehnungslehre* was obscurely described and needed interpretation, such as it received from Victor Schlegel in the years 1872–75; even after that it spread very slowly and found little use in applied mathematics.

In America quaternions engaged the attention of mathematicians at Harvard University and Yale University. Gibbs of Yale proceeded to modify the quaternion system. Like Arthur Cayley and William Thomson (Lord Kelvin) in Great Britain, he felt that quaternions were not as serviceable a tool of research as had been claimed. Gibbs rejected the quaternion concept, introduced two products of vectors in the place of Hamilton's one product, and built up a system called "vector analysis." Gibbs's was the first important abandonment of Hamilton's great creation. In 1881 Gibbs[1] began to issue his *Elements of Vector Analysis, arranged for the Use of Students of Physics* (New Haven, 1881–84) (lithographed issue) in which he introduced his notation $\alpha.\beta$, $\alpha\times\beta$, $\alpha\times\beta.\gamma$, $(\alpha.\beta)\gamma$, $\alpha\times[\beta\times\gamma]$, using small Greek letters for vectors and small Latin letters for scalars. Tait in the Preface to the third edition of his *Elementary Treatise on Quaternions* opened a controversy by the statement that Gibbs "must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on *Vector Analysis;* a sort of hermaphrodite monster, compounded of the notations of Hamilton and Grassmann." Gibbs made reply,[2] but the controversy converted neither side to the views of the other. In 1901 there was published E. B. Wilson's textbook on *Vector Analysis*, founded on the lectures of Gibbs.[3] It contained Gibbs's notation $\mathbf{a}.\mathbf{b}$ for the scalar product and $\mathbf{a}\times\mathbf{b}$ for the vector product.

It is unfortunate that no uniform notation has yet been reached. If the question were simply one involving the products of two vectors, it would not matter very much which notation was adopted. But the issue is more complicated. In the first place, what definitions for the product are the most useful in the applications of vector analysis? What concepts should be included in a minimum system? These matters cannot be considered here. In the second place, in the products of

[1] See *Scientific Papers of J. Willard Gibbs,* Vol. II (1906), p. 17–90.

[2] *Nature,* Vol. XLIV (London, 1891), p. 79–82.

[3] *Vector Analysis—Founded on the Lectures of J. Willard Gibbs* . . . . by Edwin Bidwell Wilson (New York, 1901).

three or more vectors, what notation will indicate in the simplest manner the kinds of products to be taken? What notation is not only free from ambiguity, but also least likely to be misunderstood? There has been much discussion of these topics. There have been those who hold that W. R. Hamilton's quaternions offered, all things considered, the best product and the best notation. Others preferred the American notation introduced by Gibbs. German and Italian mathematicians have different designations of their own.

509. *Attempts at unification.*—The disagreements of mathematicians on the value of quaternions and on the best substitute for quaternions led in 1895 to the first step toward the organization of an International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics, which had among its presidents Charles Jasper Joly, professor of astronomy at Dublin, and Alexander Macfarlane, at one time professor at the University of Texas. Valuable discussions[1] were carried on, but no agreements on theoretical points and on notations were reached by this body. The death of Joly and Macfarlane, and the outbreak of the great war stopped the activities of the Association.

A second attempt at unification was started in 1903 when L. Prandtl of Göttingen sent out a circular suggesting certain rules for a uniform notation. In the main, he favored Gibbs's notation. In September, 1903, he read a paper on this subject at the Naturforscherversammlung at Kassel, and Felix Klein appointed a committee to investigate and report. Articles appeared in the *Jahresbericht*[2] by L. Prandtl, R. Mehmke, and A. Macfarlane. Professor Klein described the results as follows: "Its members could not fully agree and although each had the generous impulse of departing from his original position and take a step toward the position of the others, the outcome was practically this, that about three new notations were originated!"[3]

A third attempt at unification was planned for the fourth International Congress of Mathematicians that was held at Rome in 1908. But the Congress limited itself to the appointment of a Commission which should report at the International Congress to be held at

[1] See A. Macfarlane, *International Association for Promoting the Study etc.* (April, 1905), p. 22.

[2] *Jahresbericht d. d. Mathematiker Vereinigung*, Vol. XIII (1904), p. 36, 217, 229; Vol. XIV (1905), p. 211.

[3] F. Klein, *Elementar-mathematik vom höheren Standpunkte aus*, Teil I (Leipzig, 1908), p. 157.

Cambridge in 1912. The Commission as a whole was somewhat inactive, and the desired unification was not reached.

As a preparatory movement for the Congress at Rome in 1908, Burali-Forti and Marcolongo published a preliminary study,[1] partly historical, setting forth what had been done and what they thought should be done. Their notations for the scalar and vector products, $a \times b$, $a \wedge b$, were a departure from the notations of previous authors, except that Grassmann in one of his papers had used $a \times b$ for the scalar product. The critics of the Italian proposal complained that these "unifiers" introduced still greater diversity. Henri Fehr of Geneva invited the leaders in the advancement of vector analysis to publish their views in *L'Enseignement mathématique*, published in Paris.[2] H. E. Timerding[3] of Strasbourg thought that characters not usually found in printing establishments should be avoided; he would use only very small characters $\times$ for the internal product, and $\wedge$ for the external product. He disapproves of "mod a" for the length of a vector, considers $|a|$ confusing, and argues in favor of introducing the notion of a *bivector* in the elements of vector analysis. Timerding goes contrary to Combebiac of Bourges in emphasizing the need of a universal notation: "The present discord in vectorial terminology is deplorable and its consequences in the development of this science are very grave, for it renders extremely difficult the study of the various researches in vector analysis and the numerous applications." E. B. Wilson[4] of Boston stated that in vector analysis absolute uniformity was undesirable, because of the very great variety of its applications. Peano[5] of Turin favored the Italian symbols for the internal and external products, but like Timerding would use small symbols (small symbols tend to *unite*, large ones to *separate* the entities); Peano would discard $|a|$ and modify mod a by writing either ma or the Hamiltonian Ta ("tensor a"); parentheses, being used in arithmetic, should be avoided in denoting the internal and external products. In the

[1] C. Burali-Forti and R. Marcolongo in *Rendiconti del Circolo Matematico di Palermo*, Vol. XXIII, p. 324 (1907); Vol. XXIV, p. 65, 318 (1907); Vol. XXV, p. 352 (1908); Vol. XXVI, p. 369 (1908).

[2] A condensed account of the Italian recommendations, with the symbols placed in tabular arrangement, was published in *L'Enseignement mathématique*, Vol. XI (1909), p. 41.

[3] H. E. Timerding, *L'Enseignement mathématique*, Vol. XI (1909), p. 129–34.

[4] E. B. Wilson, *L'Enseignement mathématique*, Vol. XI (1909), p. 211. See also *Bull. Amer. Math. Soc.* (2d ser.), Vol. XVI (1910), p. 415–36.

[5] G. Peano, *op. cit.*, Vol. XI (1909), p. 216.

function $f(x)$, the parenthesis is useless; it would be well to return to the Lagrangian fx. Knott[1] of Edinburgh preferred the complete product of two vectors, as given by W. R. Hamilton; he says that the ∇ of Hamilton and Tait is only partially represented in the system of Gibbs by ∇, $\nabla.$, $\nabla\times$ and in other systems by *grad*, *div.*, *rot.* Knott maintains that Hamilton accomplished with four symbols S, V, ∇, ϕ what in the minimum system proposed by the Italians requires nine or eleven symbols $\times$, $\wedge$, *grad.*, *rot*, *div*, Δ, Δ', α, *grad* α, $\frac{du}{dP}, \frac{d\mathbf{u}}{dP}$. Knott and Macfarlane[2] did not consider the Italian program superior to that of Gibbs.

Thus, a deadlock was reached on the subject of vectorial notation at the time when the great war broke out; since then international endeavors at unification have not yet been resumed.[3]

510. *Tensors.*—Recent developments of electrodynamics and of the theory of relativity have led to a new calculus of directed quantity, the calculus of tensors. The tensor of the Hamiltonian quaternions was simply a numerical factor which stretched the unit vector so that it attained the proper length. Hamilton marked the tensor of the vector q by writing Tq. He says: "I employ . . . . the two capital letters T and U as characteristics of the two operations which I call *taking the tensor*, and *taking the versor* respectively, ... which satisfy the two following general equations, or symbolical identities (in the present system of symbols):

$$q = Tq \times Uq\,; \qquad q = Uq \times Tq\,.\text{"}^4$$

Other notations have been suggested by Irving Stringham,[5] such as *tsr* for "tensor of" and *vsr* for "versor of," *amp* for "amplitude of." The recent use of the word "tensor" is different; the "tensor" is itself a directed quantity of a general type which becomes the ordinary vector in special cases. Elaborate notations which, for lack of space,

[1] C. G. Knott, *op. cit.*, Vol. XII (1910), p. 39.

[2] A. Macfarlane, *op. cit.*, Vol. XII (1910), p. 45.

[3] In 1912 a very full and detailed statement of the different notations which had been proposed up to that time was made by James Byrnie Shaw in the publication entitled *International Association for the Study of Quaternions and Allied Systems of Mathematics* (Lancaster, Pa.; June 12, 1912), p. 18–33. More recent lists of symbols are found in Shaw's *Vector Calculus* (New York, 1922), p. 12, 127, 136, 165, 166, 179–81, 248–52. See also a historical and critical article by C. Burali-Forti and R. Marcolongo in *Isis*, Vol. II (1914), p. 174–82.

[4] W. R. Hamilton, *Lectures on Quaternions* (Dublin, 1853), p. 88.

[5] Irving Stringham, *Uniplanar Algebra* (San Francisco, 1893), p. xiii.

must be omitted here were proposed in 1914 by J. A. Schouten.[1] For example, he introduces ⌐ and ⌞ as signs of multiplication of vectors with each other, and ⌉ and ⌊ as signs of multiplication of affinors (including tensors) with each other, and uses fifty or more multiples of these signs for multiplications, their inverses and derivations. In the multiplication of geometric magnitudes of still higher order (Septoren) he employs the sign —o and combinations of it with others. These symbols have not been adopted by other writers.

A tensor calculus was elaborated in 1901 chiefly by G. Ricci and T. Levi-Civita, and later by A. Einstein, H. Weyl, and others. There are various notations. The distance between two world-points very near to each other, in the theory of relativity, as expressed by Einstein in 1914[2] is $ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx_\mu\, dy_\nu$, where $g_{\mu\nu}$ is a symmetric tensor of second rank (having two subscripts μ and ν) which embraces sixteen products $A_\mu B_\nu$ of two covariant vectors (A_μ) and (B_ν), where $x=x_1$, $x_2=y$, $x_3=z$, $x_4=idt$. He refers to Minkowski and Laue and represents, as Laue[3] had done, a symmetric tensor by the black-faced letter $\mathbf{p}$ and its components by $\mathbf{p}_{xx}$, $\mathbf{p}_{xy}$, $\mathbf{p}_{xz}$, $\mathbf{p}_{yx}$, $\mathbf{p}_{yy}$, $\mathbf{p}_{yz}$, $\mathbf{p}_{zx}$, $\mathbf{p}_{zy}$, $\mathbf{p}_{zz}$. A world-tensor has sixteen components. Laue denotes the symmetric tensor by the black-faced letter $\mathbf{t}$. Representing vectors by German letters, either capital or small, Laue marks the vector products of a vector and tensor by $[\mathfrak{q}\mathbf{p}]$ or $[\mathfrak{A}\mathbf{t}]$.

In writing the expression for ds^2 Einstein in other places dropped the Σ, for the sake of brevity, the summation being understood in such a case.[4] Pauli uses the symbolism $x=x^1$, $y=x^2$, $z=x^3$, $u=x^4$, writing briefly x^i, and expressed the formula for the square of the distance thus, $ds^2 = g_{ik}dx^i dx^k$, where $g_{ik}=g_{ki}$, it being understood that i and k, independently of each other, assume the values 1, 2, 3, 4. Pauli[5] takes generally the magnitudes $a_{iklm\ldots}{}^{rst\ldots}$, in which the indices assume independently the values 1, 2, 3, 4, and calls them "tensor components," if they satisfy certain conditions of co-ordinate

[1] J. A. Schouten, *Grundlagen der Vektor- und Affinoranalysis* (Leipzig, 1914), p. 64, 87, 93, 95, 105, 111.

[2] A. Einstein, *Sitzungsberichte der Königl. P. Akademie der Wissensch.*, Vol. XXVIII (1914), p. 1033, 1036.

[3] M. Laue, *Das Relativitätsprinzip* (Braunschweig, 1911), p. 192, 193.

[4] Einstein explains the dropping of Σ in *Annalen der Physik*, Vol. XLIX (1916), p. 781. See H. A. Lorentz and Others, *The Principle of Relativity, a Collection of Original Memoirs* (London, 1923), p. 122.

[5] W. Pauli, Jr., in *Encyclopädie* (Leipzig, 1921), Band V2, Heft 4, Art. V 19, p. 569.

transformation.[1] The rank of the tensor is determined by the number of indices of its components. The tensors of the first rank are also called "vectors." If

$$g_{ik} = \delta_i^k = \begin{cases} 0 \text{ for } i \neq k \\ 1 \text{ for } i = k \end{cases}$$

as in the case of orthogonal co-ordinates, then these g_{ik} are tensor components which assume the same value in all co-ordinate systems. Einstein[2] had used the notation d_μ^ν (§ 463); Eddington[3] used g_σ^ν.

A tensor of the second rank is what Gibbs and Wilson had called a "dyadic."[4] C. E. Weatherburn[5] writes $\boldsymbol{r}' = \boldsymbol{r} \bullet (\boldsymbol{ai} + \boldsymbol{bj} + \boldsymbol{ck})$, where the expression in brackets is the operator called "dyadic," and each term of the expanded $\boldsymbol{r}'$ is called a "dyad." Weatherburn uses also the capital Greek letters Φ, Ψ, Ω to indicate dyadics, and these letters with the suffix c, as in Φ_c, etc., to indicate their conjugates. Gibbs and Wilson marked a dyad by juxtaposition $\boldsymbol{ab}$; Heaviside by $\boldsymbol{a} \bullet \boldsymbol{b}$; Ignatowsky by $\mathfrak{A}; \mathfrak{B}$. These citations from various authors indicate wide divergence, both in nomenclature and notation in the tensor calculus.

In the consideration of the metrical continuum for Riemann's space, three-indices symbols have been introduced by E. B. Christoffel,[6] such that

$$\begin{bmatrix} gh \\ k \end{bmatrix} = \tfrac{1}{2}\left[\frac{\partial w_{gk}}{\partial x_h} + \frac{\partial w_{hk}}{\partial x_g} - \frac{\partial w_{gh}}{\partial x_k}\right], \quad \begin{Bmatrix} il \\ r \end{Bmatrix} = \sum_k \begin{bmatrix} il \\ k \end{bmatrix} \frac{E_{rk}}{E}.$$

Weyl[7] denotes Christoffel's three-index symbol by $\begin{bmatrix} ik \\ r \end{bmatrix}$ and $\begin{Bmatrix} ik \\ r \end{Bmatrix}$; Eddington[8] by $[\mu\nu, \lambda]$ and $\{\mu\nu, \lambda\}$; Birkhoff[9] by $\Gamma_{i,jk}$ and Γ_{jk}^i.

[1] W. Pauli, Jr., *op. cit.*, p. 572.

[2] H. A. Lorentz and Others, *The Principle of Relativity, a Collection of Original Memoirs* (London, 1923), p. 128.

[3] A. S. Eddington, *Report on the Relativity Theory of Gravitation* (1918), p. 34.

[4] R. Weitzenböck, *Encyclopädie d. Math. Wissensch.*, Band CXI, 3, Heft 6 (1922), p. 25.

[5] C. E. Weatherburn, *Advanced Vector Analysis* (London, 1924), p. 81, 82.

[6] E. B. Christoffel, *Journal f. reine u. angew. Mathematik*, Vol. LXX (1869), p. 48, 49.

[7] Hermann Weyl, *Raum-Zeit-Materie* (Berlin, 1921), p. 119.

[8] A. S. Eddington, *op. cit.* (1918), p. 36.

[9] G. D. Birkhoff, *Relativity and Modern Physics* (Cambridge, 1923), p. 114.

II

SYMBOLS IN MODERN ANALYSIS

TRIGONOMETRIC NOTATIONS

511. *Origin of the modern symbols for degrees, minutes, and seconds.*—Signs resembling those now in use are found in the *Syntaxis* (*Almagest*) of Ptolemy, where the Babylonian sexagesimal fractions are used in astronomical calculations. The units were sometimes called *μοῖραι* and frequently denoted by the abbreviation μ°. The *first sixtieths* or minutes were marked with one accent, the *second sixtieths* with two accents. Thus,[1] *μοιρῶν* $\mu\zeta$ $\mu\beta'$ μ'' stood for $47^{\circ}42'40''$; $\mu^{\circ}\beta$ stood for 2°. From these facts it would seem that our signs $^{\circ}$, $'$, $''$ for degrees, minutes, and seconds were of Greek origin. But it is difficult to uphold this view, especially for the sign $^{\circ}$ for "degrees." Such a line of descent has not been established. However, Edward O. von Lippmann[2] holds to the view that $^{\circ}$ is the Greek letter omicron taken from the word *μοῖραι*.

Medieval manuscripts and early printed books contain abbreviations of Latin words in place of the signs $^{\circ}\ '\ ''$. J. L. Walther[3] gives in his *Lexicon* abbreviations for *gradus* as they occur in early Latin manuscripts. The abbreviations are simply florescent forms of *gu*, *gdu*, *gdus*.

In Athelard of Bath's translation into Latin (twelfth century) of certain Arabic astronomical tables, the names *signa, gradus, minutae, secundae,* etc., are abbreviated. The contractions are not always the same, but the more common ones are *Sig.*, *Gr.*, *Min.*, *Sec.*[4] In the Alfonsine tables,[5] published in 1252, one finds marked by $49^{\ddot{2}}\ 32^{\ddot{1}}\ 15^{\ddot{m}}\ 4^{\ddot{2}}$ the sexagesimals $49\times 60+32+15\times\frac{1}{60}+4\times\frac{1}{60^2}$.

[1] Sir Thomas Heath, *A History of Greek Mathematics*, Vol. I (Oxford, 1921), p. 45.

[2] E. O. von Lippmann, *Entstehung und Ausbreitung der Alchemie* (Berlin, 1919), p. 353.

[3] *Lexicon diplomaticum abbreviationes .... Joannis Lvdolfi Waltheri ....* (Ulmae, 1756), p. 138, 139.

[4] See H. Suter, *Die astronomischen Tafeln des Muḥammed ibn Mūsā Al-Khowārizmī* in der Bearbeitung des Maslama ibn Aḥmed Al-Madjrītī und der latein. Uebersetzung des Athelard von Bath (Copenhagen, 1914), p. xxiv.

[5] A. Wegener in *Bibliotheca mathematica* (3d ser.), Vol. VI (1905), p. 179.

512. A different designation is found in a manuscript in one of the Munich libraries,[1] which gives a table (Fig. 116) to facilitate the formation of the product of two sexagesimal fractions. In the left-hand column are given the symbols for "degrees," "minutes," "seconds," "thirds," etc., down to "ninths." The same symbols appear at the head of the respective columns. Using this table as one would a multiplication table, one finds, for example, that "thirds" times "seconds" give "fifths," where the symbol for "fifths" is in the row marked on the left by the symbol for "thirds" and in the column marked at the head by the symbol for "seconds."[2]

The notation of Regiomontanus is explained in § 126. The 1515 edition of Ptolemy's *Almagest*,[3] brought out in Cologne, contains the signs s̈, g̈, m̃ for *signa, gradus, minutae.*

Oronce Fine[4] in 1535 used the marks *grad*, *m̃*, 2̃, 3̃. Sebastianus Munsterus[5] in 1536 had *S.*, *G.*, *m̃* for *signa, gradus, minutae*, respectively. Regius[6] in 1536 refers to the Alfonsine tables and gives the signs *T, s, g, m, s, t, qr.*, and, in an example, writes *T, S, gr, m̃i*, 2*ã*, 3*ã*, 4*ã*, 5*ã*. Here 1*T* (*tota revolutio*) equals 12*S* (*signa*), 1*S* equals 30 *grad.*, 1 *gr.* equals 60 *m̃i.*, etc. In 1541 Peter Apianus[7] wrote *ḡ.*, *m̃.*

[1] M. Curtze in *Abhandlungen zur Geschichte der Mathematik*, Vol. VIII (1898), p. 21. Algorithmus-manuscripts Clm. 13021, fol. 27–29 of the Munich Staatsbibliothek.

[2] The multiplication of 2°45′ by 3°10′30″, yielding the product 8°43′52″30‴, is given as follows: "Sint quoque verbi gratia 2 gradus et 45 minuta, quae per 3 gradus et 10 minuta ac 30 secunda ducantur. Resolvatur itaque per 60 unusquisque ad suam ultimam differentiam, id est minuta et secunda scilicet 165 et 11430, quae si inter se ducantur, erunt 1885950. Quae omnia si per 60 dividantur ad secunda reducentur, eruntque 31432 secunda, et remanebunt 30 tertia. Quae secunda eadem divisione 523 minuta remanentibus 52 secundis ostendunt, et hi itidem divisa 8 gradus indicant 43 minutis supereminentibus hoc modo: $\begin{matrix} 8 \\ 43 \\ 52 \\ 30 \end{matrix}$."

[3] *Almagestū, Cl. Ptolemei Pheludiensis Alexandrini Astronomo' principis . . . . Colonie'sis Germani* (1515), fol. 78.

[4] Orontius Finaeus, *Arithmetica practica* (Paris, 1535), p. 46. We quote from Tropfke, *op. cit.*, Vol. I (2d ed., 1921), p. 43.

[5] Sebastianvs Munstervs, *Organum Uranicum* (Basel, 1536), p. 49.

[6] Hudalrich Regius, *Vtrivsqve arithmetices epitome* (Strasbourg, 1536), fol. 81*B*, 82*B*.

[7] Peter Apianus, *Instrvmentvm sinvvm, sev primi mobilis* (Nürnberg, 1541), in the tables.

513. In the 1566 edition[1] of the great work of N. Copernicus, the *De revolutionibus*, there is given on folio 15 a table of natural sines, arranged in three columns. The first column contains the degrees and minutes (named *circūferentiae pt., sec.*). The second column gives the sines of the angles to a radius of 100000 parts (named *Semisses dup. circūferen.*, i.e., "half the chord of double the arc"). The third

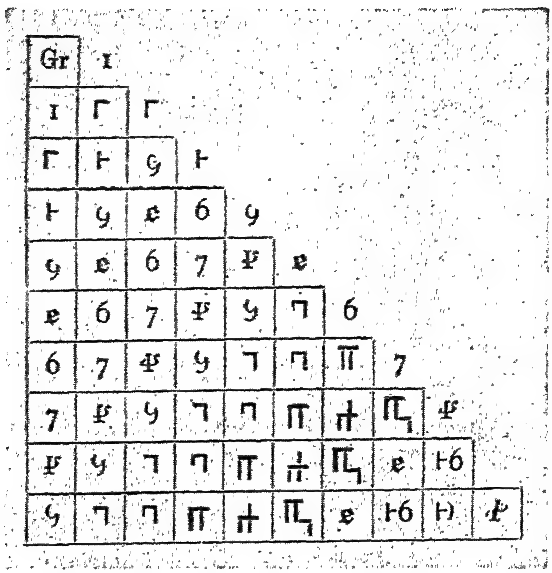


FIG. 116.—Multiplication table for sexagesimal fractions. MS in Munich Staatsbibliothek.

column contains the differences in the values of sines for angles differing by 10 minutes and has the heading *Differẽtiae* or "Differences" in the 1566 edition, but in the original edition of 1543 this column had the heading *Unius gradus partes.*[2] The sine of 30 *pt.* is given in the table as 50000. The *pt.* at the head of the first column stands for *partes,* in the text also called *gradus;* the *sec.* stands for *scrupulus.* Further on in the edition of 1566 the abbreviation *scr.* takes the place of *sec.*

Among the writers immediately following Copernicus, the Italian

[1] Nicolaus Copernicus, *De revolutionibus orbium celestium,* Liber VI (Basel, 1566).

[2] See the edition of 1873, brought out in Nürnberg, p. 44.

Gioseppe[1] uses *Gra., Mi.;* Schöner[2] writes *S., gr., m̃.* and also *S., G., M.;* Nunez[3] has *Gr., m̃.*, while Lansberge[4] has no sign for degrees and writes "23 28′ 30″." In the works of Clavius,[5] one still finds *G., M., S.*

In the foregoing publications, we have looked in vain for evidence which would connect the modern signs for circular measure directly with the Greek signs of Ptolemy. The modern signs came into use in the sixteenth century. No connection with ancient symbols has been established. We proceed to show that our modern symbols appear to be an outgrowth of the exponential concept.

514. Gemma Frisius[6] in 1540 wrote

"Integr.	*Mi.*	2.	3.	4.
36.	30.	24	50	15"

for our modern $36°30'24''50'''15^{iv}$, but in the revised 1569 edition of his book published in Paris there is an Appendix on astronomical fractions, due to J. Peletier (dated 1558), where one finds

$$\text{“Integra, } Mi \text{ vel } \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \text{ etc.”}$$
$$\circ \quad . \quad 1, 2, 3, 4, 5, 6, 7, 8.$$

This is the first modern appearance that I have found of ° for *integra* or "degrees." It is explained that the denomination of the product of two such denominate numbers is obtained by combining the denominations of the factors; minutes times seconds give thirds, because $1+2=3$. The denomination ° for integers or degrees is necessary to impart generality to this mode or procedure. "Integers when multiplied by seconds make seconds, when multiplied by thirds make thirds" (fol. 62, 76). It is possible that Peletier is the originator of the ° for degrees. But nowhere in this book have I been able to find the modern angular notation ° ′ ″ used in writing angles. The ° is used only in multiplication. The angle of 12 minutes and 20 seconds is written "*S.* 0. *g̃.* 0 *m̃.* 12. $\tilde{2}$. 20" (fol. 76).

[1] *L'Efemeridi Dim. Gioseppe, Moleto matematico. per anni XVIII* (Venice, 1563), p. 32.

[2] *Opera mathematica Joannis Schoneri Carolostadii in vnvm volvmen congesta* (Nürnberg, 1551).

[3] *Petri Nonii Salaciensis Opera* (Basel [1556]), p. 41.

[4] *Philippi Lansbergi Geometriae liber quatuor* (Leyden, 1591), p. 18.

[5] *Ch. Clavii Bambergensis Opera Mathematica* (Mayence, 1611), Vol. II, p. 20.

[6] *Gemma Frisius, Arithmeticae practicae methodus facilis* (Strasbourg, 1559), p. 57*v*°. From Tropfke, *op. cit.*, Vol. I, p. 43.

Twelve years later one finds in a book of Johann Caramuel[1] the signs $^{\circ}\ '\ ''\ '''\ ^{iv}$ used in designating angles. In 1571 Erasmus Reinhold[2] gave an elaborate explanation of sexagesimal fractions as applied to angular measure and wrote "$^{\circ}63\ '13\ ''53$," and also "$62^{\circ}\ 54'\ 18''$." The positions of the $^{\circ}\ '\ ''$ are slightly different in the two examples. This notation was adopted by Tycho Brahe[3] who in his comments of 1573 on his *Nova Stella* writes $75^{\circ}\ 5'$, etc.

As pointed out by Tropfke,[4] the notation $^{\circ}\ '\ ''$ was used with only minute variations by L. Schoner (1586), Paul Reesen (1587), Raymarus Ursus (1588), Barth. Pitiscus (1600), Herwart von Hohenburg (1610), Peter Crüger (1612), Albert Girard (1626). The present writer has found this notation also in Rhaeticus (1596),[5] Kepler (1604),[6] and Oughtred.[7] But it did not become universal. In later years many authors designated degrees by *Grad.* or *Gr.*, or *G.;* minutes by *Min.* or *M.;* seconds by *Sec.* or *S.* Thus in Horsley's edition[8] of Isaac Newton's works one finds "79 *gr.* $47'\ 48''$."

When sexagesimals ceased to be used, except in time and angular measure, subdivisions of seconds into thirds became very uncommon. In the eighteenth century the expression found in H. Scherwin's *Tables* of 1741 (p. 24), "natural Sine of $1^{\circ}\ 48'\ 28''\ 12'''$," is decidedly uncommon. The unit *S* (*signa*) occurs as late as the eighteenth century.[9]

Some early writers place the signs ', ", and also the ° when it is used, above the numbers, and not in the now usual position where exponents are placed. They would write $7^{\circ}\ 50'$ in this manner, $\overset{\circ}{7}\ \overset{1}{50}$. Among such writers are Schoner, Reesen, Ursus, and Wright (1616).

[1] *Joannis Caramvelis, Mathesis Bicepts Vetus, et Nova* (Companiae, 1670), p. 61.

[2] E. Reinhold, *Prvtenicae tabulae coelestium motuum* (Tübingen, 1571), fol. 15.

[3] Tycho Brahe, *Opera omnia* (ed. I. L. E. Dreyer), Vol. I (Copenhagen, 1913), p. 137.

[4] Tropfke, *op. cit.*, Vol. I (2d ed., 1921), p. 43.

[5] *Opus Palatinvm triangvlis a Georgio Joachimo Rhetico coeptvm* (1696), p. 3 ff.

[6] J. Kepler, *Ad vitellionem paralipomena quibus astronomiae pars optica traditur* (Frankfurt, 1604), p. 103, 139, 237.

[7] William Oughtred in *Clavis mathematicae* (1631); anonymous "Appendix" to E. Wright's 1618 translation of John Napier's *Descriptio*.

[8] S. Horsley, *Newtoni Opera*, Vol. I (1779), p. 154.

[9] Jean Bernoulli, *Recueil pour des Astronomes*, Tome II (Berlin, 1772), p. 264.

A novel mark is adopted by Georg Vega,[1] he lets $45^{\smile}$ stand for 45° but occasionally uses the full circle or zero.

The weight of evidence at hand favors the conclusion that the sign $^\circ$ for degree is the numeral zero used as an exponent in the theory of sexagesimal fractions and that it is not the Greek letter omicron.

To prevent confusion[2] between circular measure and time measure, both of which involve the words "minutes" and "seconds," and both of which are used side by side by astronomers and navigators, it became the common practice to mark minutes and seconds of arc by ′ and ″, and minutes and seconds of time by ^{m} and ^{s} (see also § 36).

515. *Sign for radians.*—The word "radian" was first used in print in 1873 by James Thomson, a brother of Lord Kelvin.[3] It has been quite customary among many authors to omit the word "radian" in the use of circular measure and to write, for example, $\frac{\pi}{2}$ or π, when $\frac{\pi}{2}$ radians or π radians are meant. It has been proposed also to use the Greek letter ρ and to write 2ρ, $\frac{3}{5}\pi\rho$ for two radians and three-fifths of π radians.[4] Others[5] have used the capital letter R in a raised position, as $\frac{\pi}{4}^R$ for $\frac{\pi}{4}$ radians, or else the small letter[6] r as in 1^r for one radian, or again the small r in parentheses[7] as in $2^{(r)}$ for two radians.

The expression "circular measure" of an angle has led to the suggestion that radians be indicated by the small letter ^{c}, i.e., 2^c means two radians. However, "when the radian is the unit angle, it is customary to use Greek letters to denote the number of radians, and the symbol ^{c} is then often omitted. When capital English letters are used, it is usually understood that the angle is measured in degrees."[8]

[1] Georg Vega, *Vorlesungen über Mathematik*, Vol. II (Vienna, 1784), p. 5, 8. 16, 23, 27, 31, 185.

[2] D. André, *op. cit.*, p. 32.

[3] Cajori, *History of Mathematics* (2d ed., 1919), p. 484.

[4] G. B. Halsted, *Mensuration* (Boston, 1881), p. 83; T. U. Taylor and C. Puryear, *Plane and Spherical Trigonometry* (Boston, 1902), p. 92.

[5] A. G. Hall and F. G. Frink, *Plane Trigonometry* (New York, 1909), p. 73.

[6] G. N. Bauer and W. E. Brooke, *Plane and Spherical Trigonometry* (Boston, 1907), p. 7.

[7] P. R. Rider and A. Davis, *Plane Trigonometry* (New York, 1923), p. 29.

[8] W. E. Paterson, *Elementary Trigonometry* (Oxford, 1911), p. 29.

Evidently there is, as yet, no approach to uniformity in the designation of radians.

In geodesy a symbol is used for a radian expressed in seconds of arc (i.e., for $180\times 60\times 60\div\pi$); Albrecht[1] uses $\frac{1}{\sin 1''}$, Helmert[2] and W. D. Lambert[3] use ρ''.

516. *Marking triangles.*—One of the devices introduced at an early period to aid the eye when one enters upon the solution of a plane or spherical triangle was the use of strokes to indicate the given parts and the parts to be computed. Perhaps the earliest author to use strokes was Philipp van Lansberge, a native of Gand. He published a trigonometry with tables, *Triangulorum geometricorum libri quator*, the dedicatory epistle of which bears the date 1591.[4] If in a spherical triangle a side is given, two parallel strokes cross it;[5] if an angle is given, the two parallel strokes appear in the angular space close to the vertex. A required part is unmarked or else marked with a single stroke.

A slight modification of these marks is found in a manuscript[6] prepared in 1599 by Melchior Jöstel, of Wittenberg, who uses one stroke for each of the given parts and a small circle for each of the required parts. The stroke is drawn so as to cross a side or bisect an angle. If an angle is required, the small circle is drawn with its center approximately at the vertex.

The use of strokes is found again in the *Trigonometria* of Bartholomaeus Pitiscus, which was published in Augsburg in 1600; later editions appeared in 1608 and 1612. An English translation was made by R. Handson. Glaisher states[7] that Pitiscus used in the 1600 edi-

[1] Theodor Albrecht, *Formeln und Hülfstafeln für geographische Ortsbestimmungen* (Leipzig, 1879; 4th ed., 1908).

[2] Helmert, *Höhere Geodäsie.*

[3] W. D. Lambert, *Effect of Variations in the Assumed Figure of the Earth on the Mapping of a Large Area*, Spec. Pub. No. 100 (U.S. Coast and Geodetic Survey, 1924), p. 35.

[4] The book is printed in P. Lansbergii, *Opera omnia* (Middelburgi Zelandiae, 1663), p. 1–88. About Lansberge see also Ad. Quetelet, *Histoire des sciences mathématiques et physiques* (Bruxelles, 1864), p. 168–79; Delambre, *Histoire de l'astronomie moderne*, Tome II (Paris, 1821), p. 40.

[5] *P. Lansbergii opera omnia*, p. 77, 85–88.

[6] Printed in *Tychonis Brahe Dani Scripta Astronomica* (ed. I. L. E. Dreyer), Vol. I (Hauniae, 1913), p. 300.

[7] J. W. L. Glaisher, *Quarterly Journal of Pure and Applied Mathematics*, Vol. XLVI (London, 1915), p. 167.

tion (p. 47) a stroke to indicate the given parts of a triangle and a small circle to indicate the required parts. This notation is the same as that of Jöstel. Oughtred owned a copy of Pitiscus as we know from John Aubrey who refers to Oughtred's book: "I myselfe have his Pitiscus, imbellished with his excellent marginall notes, which I esteeme as a great rarity." There is no doubt that Oughtred received the suggestion of using strokes and little circles from Pitiscus.

In the 1618 edition of Edward Wright's translation, from Latin into English, of John Napier's *Mirifici Logarithmorum Canonis Descriptio*, there is an anonymous Appendix bearing the title, "An Appendix to the Logarithmes," probably from the pen of William Oughtred. It is historically a remarkable document, for it is here that one first finds the $\times$ in the form of the letter X to designate multiplication, the earliest table of natural logarithms, the first appearance of the radix method of computing logarithms, the first printed appearance of $=$ as a sign of equality after the time of Recorde, and the earliest use in England of abbreviations for the trigonometric functions.[1]

This "Appendix" marks the known parts of a triangle with one stroke and the unknown parts with a small circle. These signs are also found in Oughtred's *Circles of Proportion* (1632), in Richard Delamain's *Grammelogia* (1630), in John Well's *Sciographia* (1635), in the second (1636) and later editions of Edmund Gunter's *Works*, as well as in John Seller's *Practical Navigation* (1671[?]). Glaisher, from whom we take these data, remarks that among the books that he has seen in which the stroke and the little circle are used, it is only in the "Appendix" and in Oughtred's *Circles of Proportion* that the little circle is placed just inside the triangle and not with its center in the side.

That there should appear variations in the mode of marking the known and unknown parts of a triangle is to be expected. In Albert Girard's *Tables des Sinus, etc.* (1626), a given side of a spherical triangle is marked by two strokes drawn across it, and a given angle by a small arc of a circle following the angle. A required side is not marked, but a required angle is marked by a point in its interior.

In Briggs's *Trigonometria Britannica* (1633), one finds the use of a dotted line to denote the required parts, the full stroke being used for

[1] The "Appendix" in question is reprinted in different parts of an article by J. W. L. Glaisher in the *Quarterly Journal of Pure and Applied Mathematics* (London, 1915), p. 125–97. The "Appendix" will be found in proper sequence on p. 140–41, 148–52, 194–96, 141–43, 192–94.

the given parts, as in the previously named publications. This modified designation is employed also in Vlacq's *Trigonometria artificialis* (1633, 1665, 1675), in Gellibrand's *Institution Trigonometricall* (1635), in John Newton's *Trigonometria Britannica* (1658), and in other publications.[1]

We shall not trace this notation through the later centuries. While it does not appear regularly in textbooks of more recent time, some symbols for marking the given parts and the required parts have been and are still widely used in elementary teaching.

517. *Early abbreviations of trigonometric lines.*—An early need for contracted writing in trigonometry is seen in Francisco Maurolyco, of Messina, in Italy, who, in a manuscript[2] dated 1555 writes "sinus $2^{\underset{\cdot}{u}s}$ arcus." In his printed edition of the *Menelai Sphaericorum libri tres* (1558),[3] he writes "sinus 1^m arcus" for "sinus rectus primus" (our sine), and also "sinus 2^m arcus" for "sinus rectus secundus" (our cosine).

Perhaps the first use of abbreviations for the trigonometric lines goes back to the physician and mathematician, Thomas Finck, a native of Flensburg in Schleswig-Holstein. We premise that we owe to him the invention in 1583 of the words "tangent" and "secant."[4] These terms did not meet with the approval of Vieta, because of the confusion likely to arise with the same names in geometry. Vieta[5] called the trigonometric tangent *Prosinus* and the trigonometric secant *Transsinuosa*. But Vieta's objection was overlooked or ignored. The trigonometric names "tangent" and "secant" were adopted by Tycho Brahe in a manuscript of 1591, by G. A. Magini in 1592, by Thomas Blundeville in 1594, and by B. Pitiscus in 1600. The need for abbreviations arose in connection with the more involved relations occurring in spherical trigonometry. In Finck's Liber XIV, which relates to spherical triangles, one finds proportions and abbreviations as follows: "sin.," "tan.," "sec.," "sin. com.," "tan. com.," and "sec. com." The last three are our cosine, cotangent, and cosecant. He gives the proportions:

[1] J. W. L. Glaisher, *op. cit.*, p. 166, 167.

[2] B. Boncompagni *Bullettino*, Vol. IX (1876), p. 76, also p. 74.

[3] A. von Braunmühl, *Bibliotheca mathematica* (3d ser.), Vol. I (1900), p. 64, 65.

[4] *Thomae Finkii Flenspurgensis Geometriae Rotundi Libri XIIII* (Bâle, 1583). All our information relating to this book is drawn from J. W. L. Glaisher's article in *Quarterly Journal of Pure and Applied Mathematics*, Vol. XLVI (1915), p. 172.

[5] Fr. Vieta, *Opera mathematica* (Leyden, 1646), p. 417, in the *Responsorum*, Liber VIII. The *Responsorum* was the first published in 1593.

(P. 360)	Rad.	sin. *ai.*	sin. *a.*	sin. *ie.*
(P. 364)	Sin. *yu*	sin. *ys*	sin. *iu*	sin. *it*
(P. 378)	Rad.	tan. com. *a.*	tan. com. *i*	sin. com. *ia.*
(P. 381)	Rad.	sin. com. *ie.*	sec. *ia.*	sec. *ea.*

It will be observed that the four terms of each proportion are written one after the other, without any symbols appearing between the terms. For instance, the first proportion yields sin ie = sin $ai \cdot$ sin a. An angle of a spherical triangle is designated by a single letter, namely the letter at the vertex; a side by the letters at its extremities. Accordingly, the triangle referred to in the first proportion is marked *a i e.*

Glaisher remarks that Finck is not at all uniform in his manner of appreviating the names of the trigonometric lines, for one finds in his book also "sin. an.," "sin. ang.," "tang.," "sin. comp.," "sin. compl.," "sin. com. an.," "tan. comp.," etc. Finck used also "sin. sec." for *sinus secundus*, which is his name for the versed sine. The reader will observe that Finck used abbreviations which (as we shall see) ultimately became more widely adopted than those of Oughtred and Norwood in England, who are noted for the great emphasis which they laid upon trigonometric symbolism.

Similar contractions in treating spherical triangles are found not long after in Philipp van Lansberge's *Triangulorum geometricorum libri quator* (1591), previously referred to. In stating proportions he uses the abbreviations[1] "sin.," "sec.," "tang.," "sin. cōp.," "tang. comp.," "tang. compl.," "sec. comp.," but gives the names in full whenever there is sufficient room in the line for them. For instance, he writes in a case of a right spherical triangle (p. 87), "ut radius ad sinum basis ita sec. comp. lat. ad sec. comp. ang."

A few years later, one finds a few abbreviations unrelated to those of Finck, in a manuscript of Jöstel, of Wittenberg, whom we have mentioned earlier. He was a friend of Longomontanus and Kepler. In 1599 he writes "S." for *Sinus*,[2] as does also Praetorius of Altdorf.

Later still one encounters quite different symbols in the *Canon triangulorum* of Adriaen van Roomen (Adrianus Romanus), of the Netherlands. He designates the sine by "S.," the tangent (which he called, as did Vieta, the *Prosinus*) by "P.", the secant or *Transsinuosa* by "T." The co-functions are indicated by writing cℓ. after the function. A rectangle or the product of two factors is indicated by

[1] Ph. van Lansberge, *op. cit.*, p. 85–88.

[2] *Tycho Brahe Dani Opera Omnia* (ed. J. L. E. Dreyer; Hauniae, 1913), Vol. I, p. 297 f.; A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, 1. Teil (Leipzig, 1900), p. 230 n.

"§," followed by "sub." Thus Romanus writes our equation $1:(\sin b \operatorname{cosec} A)=(\cot c \div \cot b \cos A):\cot C$ in the following ponderous manner:[1]

"Ut quadratum radii ad § sub S. *AB* & T. ȼ *A*,

Ita § sub P. ȼ { *AC* & Radio / *AB* & S. ȼ *A* } ad § sub Radio & / Prosinus ȼ *C* } efficiente / coefficiente."

During the next quarter-century the leading experiments on trigonometric symbolism were carried on in Great Britain, where the invention of logarithms gave a tremendous impetus to trigonometric computation.

518. *Great Britain during 1602–18.*—We begin with Nathaniel Torporley's trigonometry, as found in his *Diclides Coelometricae seu valvae astronomicae universales* (London, 1602). Torporley, who had been at Christ Church, Oxford, is said to have been amanuensis to Vieta in France and to have conveyed ideas of Vieta to Thomas Harriot. De Morgan[2] describes Torporley's book as follows: "Torporley has given two tables of double entry, which Delambre says are the most obscure and incommodious that ever were made. The first is neither one nor the other; a and b being the arguments, and c the tabulated result, it amounts to $\tan c=\tan a \times \sin b$, the double entry being contrived like that of the common multiplication table. Of the second table, as the book is scarce, I subjoin half a dozen instances.

30°

	G	*G*	*M*	*M*
24°	21		27	
		8		33
	12		54	

54°

	G	*G*	*M*	*M*
70°	27		55	
		26		5
	1		50	

70°

	G	*G*	*M*	*M*
54°	39		14	
		30		46
	8		28	

90°

	G	*G*	*M*	*M*
19°	71		58	
		18		2
	53		56	

19°

	G	*G*	*M*	*M*
90°	9		30	
		9		30
	0		0	

147°

	G	*G*	*M*	*M*
59°	88		3	
		58		57
	29		6	

[1] *Adriani Romani Canon triangvlorum sphaericum* (Mayence, 1609), p. 121. See also A. von Braunmühl in *Bibliotheca mathematica* (3d ser.), Vol. I (1890), p. 65, who refers to p. 230 of the *Canon triangvlorum.*

[2] A. de Morgan, "On the Invention of Circular Parts," *Philosophical Magazine,* Vol. XXII (1843), p. 352.

As far as the formulae for right-angled [spherical] triangles are concerned, this table applies as follows. The sine of the angle on the left multiplied by the sine of the upper angle in the square compartment, gives the sine of the second angle in that compartment. Thus Torporley means to say that

$$\sin 24° \times \sin 21°27' = \sin 8°33' .$$

Those who like such questions may find out the meaning of the other parts of the table. . . . . Torporley was an astrologer."

The evidence at our command does not indicate that English mathematicians of the early part of the seventeenth century had seen the trigonometric work of Finck, Lansberge, and Romanus on the Continent. Some English authors were familiar with the very influential work of Pitiscus, but he did not use abbreviations for the trigonometric lines. The earliest efforts in the way of improved notation made in England occur in that brief but remarkable anonymous "Appendix" in the 1618 edition of Edward Wright's translation of Napier's *Descriptio*. We quote the following from this "Appendix":

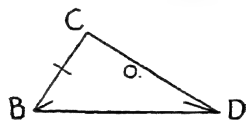


FIG. 117.—Marking the given and required parts of a triangle (1618)

"THE CALCULATION OF A PLAINE OBLIQUE ANGLED TRIANGLE

"1. Either the foure ingredient parts are opposite two to two: as

$$sB+BC=sD+DC .$$

"2. Or the two sides being given with the angle comprehended within them, either of the other two angles is sought.

The angle B is 37°. 42′ .
The side BC is 39 .
& the side BD is 85 ."[1]

Parts of the triangle are marked in the manner previously noted. The equation by which the unknown side DC is to be computed involves the Law of Sines. It is clear that the four terms are intended to represent *logarithmic* numbers; that is, sB means really log sin B; BC stands for log (BC), etc. If such were not the intention, the two signs of addition + would have to be replaced each by the sign

[1] See *Quarterly Journal of Pure and Applied Mathematics*, Vol. XLVI (1915), p. 150.

of multiplication, $\times$. The abbreviations of trigonometric lines are "s" for "sine" and "t" for "tangent," "s_*" (in one place "s*") for "cosine," and "t_*" for "cotangent." Note that the author writes "s," not "s."; there are no dots after the abbreviations. This "Appendix" is the earliest book which contains the abbreviations of the words "sine" and "cosine" used as constituents of formulae other than proportions. In the writing of equations, the sign of equality (=) is used, perhaps for the first time in print sine 1557 when Robert Recorde introduced it in his *Whetstone of Witte.* Proportions were presented before 1631 in rhetorical form or else usually by writing the four terms in the same line, with a blank space or perhaps a dot or dash, to separate adjacent terms. These proportions would hardly take the rank of "formulae" in the modern sense. Another peculiarity which we shall observe again in later writers is found in this "Appendix." Sometimes the letter *s* in "s_*" (cosine) is omitted and the reader is expected to supply it. Thus the equation which in modern form is

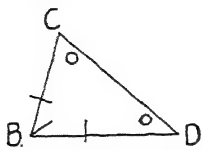


FIG. 118.—The given and the required parts of another triangle (1618)

$$\log\cos\overline{BA}+\log\cos\overline{CA}=\log\cos\overline{BC}$$

is given in the "Appendix" in extremely compact form as

$$s_*BA+{}^*CA={}_*BC\ ,$$

the *logarithmic* numbers being here understood.

As regards the lettering of a triangle, the "Appendix" of 1618 contains the following: "It will bee convenient in every calculation, to have in your view a triangle, described according to the present occasion: and if it bee a right angled triangle, to denote it with the Letters *A.B.C:* so that *A* may bee alwayes the right angle; *B* the angle at the Base *B.A* and *C* the angle at the Cathetus *CA*."[1] The "cathetus" is the vertical leg of the right triangle. J. W. L. Glaisher makes the following observation: "It is evident that *A* is taken to be the right angle in the right-angled triangle *ABC* in order that *BA* may represent the base and *CA* the cathetus, the letters indicating the words."[2] An oblique triangle is marked *BCD* with *BD* as the base, so that the perpendicular (or cathetus) drawn from the vertex to the base could be marked *CA*. The fact that this same unique mode of

[1] *Quarterly Journal of Mathematics,* Vol. XLVI (1915), p. 162.

[2] *Op. cit.,* p. 162.

lettering is employed by Oughtred in the books which bear his name is one of several arguments in support of the view that Oughtred is the author of the "Appendix."

519. *European Continent during 1622–32.*—Meanwhile, some attention to symbolism continued to be given on the Continent. The Danish astronomer, C. S. Longomontanus,[1] in 1622 used the notation "S.R." for *sinus rectus* (sine), "S.T." for *sinus totus* (i.e., sin 90° or radius), "T." and "Tang." for *tangens* (tangent), "Sec." for *secans* (secant), "T. cōpl." for "cotangent," "Sec. Compl." for "cosecant."

In 1626 a work was published at the Hague by Albert Girard of Lorraine with the title *Tables de sinus, tangentes et sécantes selon le raid de 10000 parties*, of which a translation into Dutch appeared at

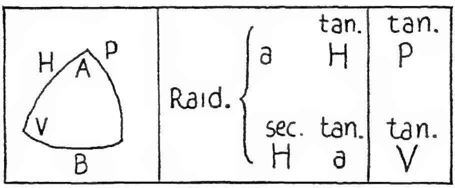


FIG. 119.—Illustrating Girard's notation in trigonometry

the Hague in 1629.[2] Girard uses in his formulae for right-angled spherical triangles H, P, B to represent the hypothenuse, perpendicular, and base, respectively, A to represent the angle at the vertex subtended by the base, V the angle at the base subtended by the perpendicular; a small letter a denotes the complement of capital A, i.e., $a=90°-A$. By a single letter is meant its sine; if a tangent or secant is intended, the abbreviation for the function is given. Thus Girard writes the formula shown in Figure 119, which means in modern notation

$$\tan P=\sin\,(90°-A)\tan H\ ,$$
$$\tan V=\sec H\tan\,(90°-A)\ .$$

When Girard uses brackets he means that the sine of the quantity included shall be taken. Thus, remembering that $b=90°-B$, etc.,

[1] Christian Longomontanus, *Astronomiae Danicae* (Amsterdam, 1622), Pars prior, p. 12 ff.

[2] Our information is drawn from J. W. L. Glaisher, *op. cit.*, Vol. XLVI (1915), p. 170–72; M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. II (2d ed.; Leipzig, 1913), p. 708, 709; Vol. III (2d ed., 1901), p. 559; A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Vol. I (Leipzig, 1900), p. 237.

one sees that $(A+b)-(d)$ signifies sin $(A+90°-B)-$sin $(90°-\mathrm{D})$, but in three cases he expresses the sine by writing the abbreviation "sin." above the angle or side, so that $\overset{\text{sin.}}{CD}$ stands for sin CD.

In further illustration we cite the following proportion from the 1629 edition, folio K_9V°, which applies to the spherical triangle ABD having AC as a perpendicular upon BD:

$$\overset{\text{sec.}}{BAC} \quad \overset{\text{sec.}}{DAC} \quad \overset{\text{tan.}}{AB} \quad \overset{\text{tan.}}{AD}\,.$$

In an Appendix of the 1629 edition of the *Tables*, "Tan. B" occurs twice, the "Tan." being written before the B, not above it.

The Dutch mathematician, Willebrord Snellius,[1] did not place much emphasis on notation, but in one place of his trigonometry of 1627 he introduces "sin.," "tang.," "sec.," "sin. com.," "tan. com.," "sin. vers.," "tan. com.," "sec. com." Here "comp." means "complement"; hence "sin. com." is our "cosine," etc.

A follower of Girard was J. J. Stampioen, who in his edition of van Schooten's *Tables of Sines*[2] gave a brief treatment of spherical trigonometry. Like Girard, he uses *equations* rather than *proportions*, but stresses symbolism more than Girard. Stampioen writes one of his formulae for an oblique spherical triangle ABC thus,

$$\frac{(:BC)\div(AB\mp AC)\text{ in }\square R}{|AB,\text{ in }AC\,|}=(:A)\,,$$
$$\text{—Sin—}$$

which means

$$\frac{\text{vers }BC-\text{vers }(AB-AC)}{\sin AB\cdot\sin AC}=\text{vers }A\,.$$

520. *Great Britain during 1624–57.*—More general was the movement toward a special symbolism in England. Contractions similar to those which we found in the 1618 "Appendix" came to be embodied in regular texts in the course of the next dozen years. In 1624 "sin" for "sine" and "tan" for "tangent" were placed on the drawing representing Gunter's scale, but Gunter did not otherwise use them in that

[1] *Willebrordi Snelli a Royen R. F. Doctrinae triangvlorvm canonicae liber qvatvor* (Leyden, 1627), p. 69.

[2] *Tabulae Sinuum . . . . door Fr. van Schooten gecorrigeert . . . . door J. J. Stampioen* (Rotterdam, 1632). See A. von Braunmühl, *Bibliotheca mathematica* (3d ser.), Vol. I (1900), p. 67.

edition of the book;[1] in the 1636 edition, a Latin passage is interpolated which contains *Sin.*, *Tan.*, *Cosi.* Noteworthy is the abbreviation *cosi* to represent the new word "cosine." To him we owe the manufacture of this word out of "sine of the complement," or "complemental sine,"[2] which he introduced in his *Canon triangulorum* (London, 1620), along with the word "cotangent"; but at that time he did not use abbreviations. Glaisher[3] says the two words "cosine" and "cotangent" did not find ready acceptance, that they were not used by Briggs, Speidell, Norwood, nor by Oughtred, but that they occur in Edmund Wingate's *Use of Logarithmes* (London, 1633), in Wells' *Sciographia* (London, 1635), in John Newton's works, and in John Seller's *Practical Navigation* (1671[?]).

More generous in the use of symbols was a teacher of mathematicians in London, John Speidell,[4] who published in 1627 a treatise on spherical triangles in which he uses "Si.," "Si. Co.," "T.," "T. Co.," "Se.," "Se. Co.," for "sine," "cosine," "tangent," "cotangent," "secant," "cosecant." As in Thomas Finck, so here, there are given abbreviations for six trigonometric lines; as in Finck, the abbreviations have a number of variants. With Speidell there is no regularity in the printing as regards capitals and small letters, so that one finds "Co." and "co.," "T. Co." and "T. co.," "Se. Co." and "se. co." Moreover, he generally leaves out the symbol "Si.," so that "Co.," the abbreviation for "complement," stands for "cosine." He writes,

$$\text{Si. } AB - \text{Si. } AD - T.\ BC - T.\ DE$$
$$90. - \text{Co.}\overset{d.}{60}.\overset{m.}{27} - \text{Se. Co.}\overset{d.}{42}\cdot\overset{m.}{34}\cdot\tfrac{1}{3}\,.$$

which means

$$\text{Sin } AB : \sin AD = \tan BC : \tan DE\ ,$$
$$\sin 90^\circ : \cos 60^\circ 27' = \operatorname{cosec} 42^\circ 34\tfrac{1}{3}' : \cos x\ .$$

Conspicuous among writers on our subject was Richard Norwood, a "Reader of Mathematicks in London," who brought out in 1631 a widely used book, his *Trigonometrie.* Norwood explains (Book I, p. 20): "In the examples, *s* stands for *sine: t* for *tangent:* *ſc* for *sine*

[1] E. Gunter, *Description and Use of the Sector, The Crosse-staffe and other Instruments* (London, 1624), Book II, p. 31.

[2] *Quarterly Journal of Mathematics*, Vol. XLVI (1915), p. 147, 187, 190.

[3] *Op. cit.*, p. 190.

[4] *A Breefe Treatise of Sphaericall Triangles.* By John Speidell, Professor and teacher of the Mathematickes in Queene-streete (London, 1627), p. 15, 29, 37.

complement: tc for *tangent complement: ſec:* for *secant.*" It is to be observed that these abbreviations are not followed by a period. As a sample, we give his arrangement of the work (p. 21) in solving a plane triangle ABD, right-angled at B, the side A and the angles being given and DB required:

"s, D 67 deg. 23′.	comp:ar:	0,0347520
AB 768 paces.		2,8853612
s, A 22–37.		9,5849685
DB 320 paces.		2,5050817"

By "comp: ar:" is meant the "arithmetical complement." Further on in his book, Norwood uses his abbreviations not only in examples, but also in the running text. In formulae like (Book II, p. 13)

$$\text{sc } A+\text{Rad}=\text{tc } AG+\text{tc } AD\ ,$$

which applies to a spherical triangle ADG having DG as a quadrant, each term must be taken as representing logarithmic values.

Oughtred, who is probably the author of the "Appendix" of 1618, uses the notation "s" and "s co" in an undated letter to W. Robinson.[1] In this letter, Oughtred uses such formulae as

$$\frac{\text{sco. } PS\times\text{sco. } PZ}{\text{sco. } ZS}=\text{s. arcus prosthaph.}$$

and

$$2s.\tfrac{1}{2}\begin{Bmatrix}Z\\X\end{Bmatrix}+\text{sco. } ZS+0.3010300(-2R)$$
$$-\text{s. } PS-\text{s } PZ(+2R)=\text{sco. } \angle P\ .$$

It is noticeable that in this letter he writes "s" and "sco" for natural and logarithmic sines and cosines indifferently, and passes from natural to the logarithmic, or vice versa, without remark.

In his *Clavis mathematicae* of 1631 (1st ed.), Oughtred does not use trigonometric lines, except the *sinus versus* which on page 76 is abbreviated "*sv.*" But the following year he brought out his *Circles of Proportion* which contain "sin" and "s" for "sine," "tan" and "t" for "tangent," "sec" and "se" for "secant," "s co" for "cosine," "tco" for "cotangent," the dot being omitted here as it was in Norwood and in the "Appendix" of 1618. The *Circles of Proportion* were brought out again at London in 1633 and at Oxford in 1660. In 1657 Oughtred published at London his *Trigonometria* in Latin and

[1] S. P. Rigaud, *Correspondence of Scientific Men of the 17th Century* (Oxford, 1841), Vol. I, p. 8–10.

also his *Trigonometrie* in English. Contractions for the trigonometric lines occur in all these works, sometimes in the briefest possible form "s," "t," "sco," "tco," at other times in the less severe form[1] of "*sin*," "*tan*," "*sec*," "*tang*." In his texts on trigonometry he uses also "s ver" for *sinus versus* ("versed sine"). In both the Latin and English editions, the abbreviations are sometimes printed in roman and sometimes in italic characters.

521. *Seventeenth-century English and continental practices are independent.*—Attention to chronological arrangement takes us to France where Pierre Hérigone brought out his six-volume *Cursus mathematicus* in 1634 and again in 1644. The entire work is replete with symbols (§ 189), some of which relate to trigonometry. It will be remembered that "equality" is expressed by 2|2, "minus" by $\sim$, "ratio" by π, "square" by $\square$, "rectangle" by ▭, "multiplication" by (ʾ). Small letters are used in the text and capitals in the accompanying figures. If in a plane triangle *abc* a perpendicular *ad* is drawn from the vertex *a* to the side *bc*, then $\cos b = \sin \underline{|bad}$. With these preliminaries the reader will recognize in the following the Law of Cosines:

$$2 \text{▭}\, ab,\, bc \;\pi\; \square\, bc + \square\, ba \sim \square\, ac \;2|2\; \text{rad.} \;\pi\; \sin < bad\,.$$

Hérigone obtains a compact statement for the Law of Tangents in the plane triangle *fgh* by assuming

$$a\ 2|2\ hf+hg,\ b\ 2|2\ hf-hg,\ c\ 2|2\ \tfrac{1}{2}, f+g\,,$$

and considering the special case when

$$\tfrac{1}{2}\,(f-g)=8°43'\,.$$

His statement is:

$$a\ \pi\ \text{tangent}\ .c\ \ 2|2\ \ b\ \pi\ \text{tangent}.8g\ 43'\,.$$

We have seen that on the Continent abbreviations for the trigonometric lines were introduced earlier than in England by Thomas Finck and a few others. In England abbreviations occur as early as 1618 but did not become current until about 1631 or 1632 in the writings of Norwood and Oughtred. But they did not become universal in England at that time; they are absent from the writings of Briggs, Gellibrand, Blundeville,[2] Bainbridge (except occasionally when the

[1] For the exact references on the occurrence of each abbreviation in the various texts, see §§ 184, 185.

[2] *Mr. Blundevil His Exercises* . . . . (7th ed.; London, 1636), p. 99.

printer is crowded for space and resorts to "Tan." and "Sin."),[1] Wingate and Wells.[2] But these writers belong to the first half of the seventeenth century. Later English writers who abstained from the use of abbreviations in trigonometric names were Collins[3] and Forster.[4]

In the latter part of the seventeenth century trigonometric symbolism reached a wider adoption in England than on the Continent. The example set by Finck, Romanus, Girard, and Snellius did not find imitators on the Continent. Thus in the *Trigonometria* of B. Cavalieri (1643) no contractions appear. In A. Tacquet's *Opera mathematica* (Antwerp, 1669) one finds in the part entitled *Geometria practica* the names of the trigonometric lines written out in full. In Kaspar Schott's *Cursus mathematicus* (Würtzburg, 1661), the same practice is observed, except that in one place (p. 165) "sin." is used. At that time another difference between the English writers and those on the Continent was that, in abbreviating, continental writers placed a dot after each symbol (as in "sin."), while the English usually omitted the dot (as in "sin").

522. *England during 1657–1700.*—In England the abbreviations came to be placed also on Gunter's scales and on slide rules. To be sure, in the earliest printed explanation of a slide rule found in Richard Delamain's *Grammelogia* (London, 1630–33[?]), the designations on the slide rule (which was circular) were selected without reference to the names of the trigonometric lines represented. Thus, the "Circle of Tangents" was marked S on the fixed circle and Y on the movable circle; the "Circle of Sines" was marked D on the fixed circle and TT on the movable circle. Seth Partridge, in his *Instrument called the Double Scale of Proportion* (London, 1662), gives no drawings and uses no abbreviations of trigonometric lines. But in Wil. Hunt's *Mathematical Companion* (London, 1697) there is a description of a slide rule having a "Line of Sines" marked S, a "Line of Artificial Sines" marked SS, a "Line of Tangents" marked T.

Figure 120 shows a page[5] in Isaac Newton's handwriting, from Newton's notebook, now in the Pierpont Morgan Library, New York. This page was probably written about the time when Newton entered Cambridge as a student.

[1] *Cl. V. Johannis Bainbrigii Astronomiae .... canicvlaria* (Oxford, 1648), p. 55.

[2] *Quarterly Journal of Mathematics*, Vol. XLVI (1915), p. 160, 166.

[3] John Collins, *Description and Uses of the General Quadrant* (London, 1658); also *The Sector on a Quadrant* (London, 1658).

[4] Mark Forster, *Arithmetical Trigonometry* (London, 1690).

[5] Taken from *Isaac Newton* (ed. W. J. Greenstreet; London, 1927), p. 29.

During the latter part of the seventeenth century and during the eighteenth century the extreme contractions "s," "t," etc., continued in England along with the use of the moderate contractions "sin.," "tan.," etc. Occasionally slight variations appear. Seth Ward, in his

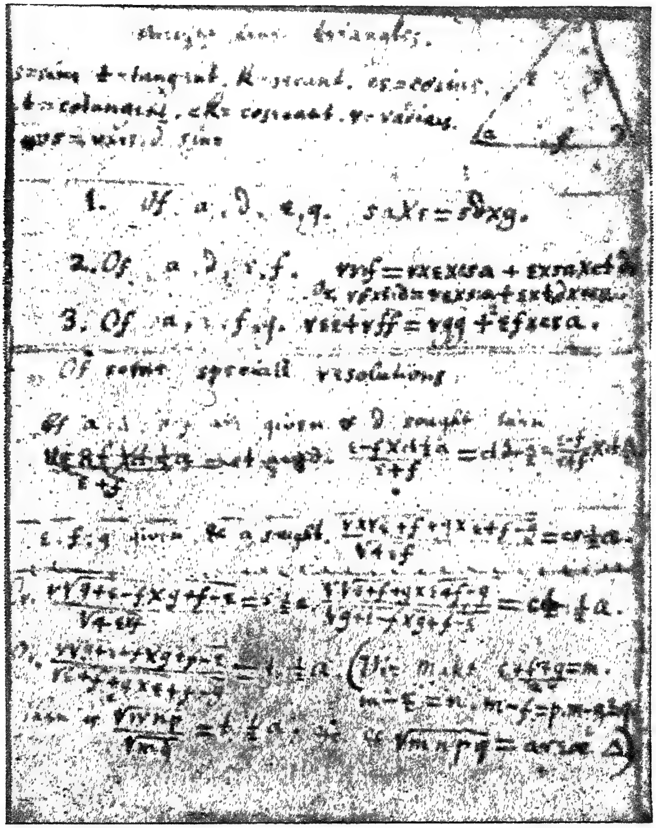


FIG. 120.—A page of Isaac Newton's notebook, showing trigonometric symbols and formulae.

Idea trigonometriae demonstratae (Oxford, 1654), used the following notation:

s. *Sinus*	' *Complementum*
s' *Sinus Complementi*	Z *Summa*
t *Tangens*	X *Differentia*
$\mathcal{T}$ *Tangens Complementi*	C *Crura*

The notation given here for "sine complement" is a close imitation of the s^* found in the anonymous (Oughtred's?) "Appendix" to E. Wright's translation of John Napier's *Descriptio* of the year 1618. Seth Ward was a pupil of Oughtred who claimed Seth Ward's exposition of trigonometry as virtually his own.

In the same year, 1654, John Newton published at London his *Institutio mathematica.* His idea seems to have been not to use symbolism except when space limitations demand it. On page 209, when dealing with formulae in spherical trigonometry, he introduces abruptly and without explanation the contractions "s," "t," "cs," "ct" for "sine," "tangent," "cosine," "cotangent." From that page we quote the following:

$$\text{“}4\left\{\begin{matrix}\text{Ra.}+\text{cs }ABF\\ \text{t}FB+\text{ct }BC\end{matrix}\right\}\quad\text{is equall to}\quad\left\{\begin{matrix}\text{ct }AB+\text{t}FB\\ \text{cs }FBC+Ra.\text{”}\end{matrix}\right.$$

"1. cs $AB+$cs FC is equall to cs $BC+$cs AF.
2. s $AF+$ct C is equall to s $FC+$ct A."

The capital letters are used in lettering the triangle and the small letters in, abbreviating the four trigonometric lines. The two are easily distinguished from each other. It is to be noted that in these formulae the "s," "cs," "t," "ct" mean the *logarithmic* lines, but on the same page they are used also to represent the natural lines, in the proportion "As cs AF, to cs FC; so is cs AB, to cs BC."

In his *Trigonometria Britannica* (London, 1658), John Newton uses the abbreviations "s," "t," "ct" in the portions of the work written by himself, but not in those translated from Briggs or Gellibrand. On pages 4 and 5 he uses "s" for "sine" in proportions, such as "Rad. DE :: s $E.DL$," and on page 66 he uses (in similar proportions) "t" and "ct" for "tangent" and "cotangent."

In the British Museum there is a copy of a pamphlet of a dozen engraved pages, prepared sometime between 1654 and 1668, by Richard Rawlinson, of Oxford, which contains "s" for "sine," "t" for "tangent," ' for "complement." But Rawlinson introduces another notation which is indeed a novelty, namely, the designation of the sides of a triangle by the capital letters A, B, C and the opposite angles by the small letters a, b, c, respectively—a notation usually ascribed to the Swiss mathematician Leonhard Euler who used a similar scheme for the first time in the *Histoire de l'académie de Berlin* (année 1753), p. 231. Rawlinson antedates Euler nearly a century, but for some reason (perhaps his obscurity as a mathematician and

his lack of assertive personality), his extremely useful and important suggestion, so very simple, failed to attract the notice of mathematicians. In his notation, A was the largest and C the smallest side of a triangle. It is interesting to see how he distinguishes between plane and spherical triangles, by writing the letters in different script. Each letter for spherical triangles was curved in its parts; each letter for plane triangles had a conspicuous straight line as a part of itself. Mathematicians as a group have never felt the need of a notation bringing out this distinction.

John Seller[1] in his *Practical Navigation* (London, 1671[?]), adopts the abbreviations for the trigonometrical lines due to Norwood. That is, he designates the cosine "sc," not "cs"; and the cotangent "tc," not "ct."

Vincent Wing[2] adopted "s.," "t.," "sec.," "cs.," "ct."; William Leybourn[3] preferred "S.," "T.," "SC.," and "TC." for "sine," "tangent," "cosine," "cotangent"; and Jonas Moore,[4] "S.," "Cos.," "T.," "Cot."

John Taylor, in his *Thesaurarium mathematicae* (London, 1687), uses "S," "Sc," "T," "Tc," "Se," "Sec," as abbreviations for "sine," "cosine," "tangent," "cotangent," "secant," "cosecant."[5] Samuel Jeake,[6] in 1696, writes "s.," "cos.," "t.," "cot.," "sec.," "cosec."

Some new symbols were introduced by John Wallis and John Caswell in a tract on trigonometry published in the 1685 and 1693 editions of Wallis' *Algebra*. Caswell's abbreviations for the trigonometric functions, as they appear in his proportions, are "S." for "sine," "Σ." for "cosine," "T" for "tangent," "τ." for "cotangent," "ʃ" for "secant," "σ" for "cosecant," "V" for "versed sine," "υ" for "1+Σ." Observe that the co-functions are designated by Greek letters. These abbreviations were introduced by John Wallis in his tract on "Angular Sections."[7] Half the perimeter of a triangle is represented by ʒ. Caswell uses also ς to represent half the sum of the sides of a triangle.

[1] See *Quarterly Journal of Mathematics*, Vol. XLVI (1915), p. 159.

[2] *Astronomia Britannica, authore Vincentio Wing, Mathem.* (London, 1669), p. 16, 48.

[3] William Leybourn, *Panorganon: or, a Universal Instrument* (London, 1672), p. 75.

[4] Jonas Moore, *Mathematical Compendium* (London, 1674). See A. von Braunmühl, *op. cit.*, Vol. II (1903), p. 45.

[5] *Quarterly Journal*, Vol. XLVI, p. 164.

[6] Samuel Jeake, ΛΟΓΙΣΤΙΚΗΛΟΓΊΑ, *or Arithmetick* (London, 1696,) p. 11.

[7] John Wallis, *Opera* (Oxford), Vol. II (1693), p. 591–92.

The expression

$$\text{“}4\,mn \cdot Z+B : \times : Z-B :: Rq \cdot \Sigma^q \tfrac{1}{2}\text{ Ang.”}$$

means

$$4\,mn:(Z+B)(Z-B)=R^2:\cos^2 \tfrac{1}{2}\text{ Angle ,}$$

where m, n, B are the sides of a plane triangle, $Z=m+n$, R is the radius of the circle defining the trigonometric functions, and the "angle" is the one opposite the side B. Another quotation:

"T 60°+T 15°=ʃ 60°=2 Rad. and *th.* 2T 60°+T 15°= T 60°+2R=T 60°+ʃ 60°=(by 1st Theorem) T 75° ,"

which signifies

$$\tan 60°+\tan 15°=\sec 60°=2\,R\text{, and therefore}$$

$$2\tan 60°+\tan 15°=\tan 60°+2R=\tan 60°+\text{sec. }60°=$$
$$\text{(by 1st Theorem) }\tan 75°\text{ .}$$

Caswell's notation was adopted by Samuel Heynes,[1] J. Wilson, J. Ward, and some other writers of the eighteenth century.

523. An English writer, Thomas Urquhart,[2] published a book in which he aimed to introduce a new universal language into trigonometry. He introduced many strange-sounding words: "that the knowledge of severall things representatively confined within a narrow compasse, might the more easily be retained in memory susceptible of their impression." This is a case of misdirected effort, where a man in the seclusion of his chamber invents a complete artificial language, with the expectation that the masses outside will accept it with the eagerness that they would gold coins dropped to them. Urquhart marks the first case, in the solution of a plane right triangle, *Uale* or *Vale;* the first two vowels, *u* or *v*, and *a* give notice of the data, the third vowel of what is demanded. A reference to the table of abbreviations yields the information that *u* means "the subtendent side," *a* means a given angle, and *e* means the required side. Hence the problem is: "Given an [acute] angle of a right triangle and the hypothenuse, to find a side." The mode of solution is indicated by the relation "Rad-U-Sapy ☞ Yr." in which "Sapy" means artificial sine of the angle opposite the side required, "U" means the logarithm of the given hypothenuse, "Yr." the logarithm of the side required. The author had one admiring friend whose recommendation follows the Preface to the book.

[1] Sam. Heynes, *A Treatise of Trigonometry* (2d ed.; London, 1702), p. 6 ff.

[2] *The Trissotetras: or, a most Exquisite Table for resolving all manner of Triangles.* . . . . By Sir Thomas Urquhart of Cromartie, Knight (London, 1645), p. 60 ff.

We close with a remark by von Braunmühl on the seventeenth-century period under consideration: "Much the same as in Germany during the last third of the seventeenth century were conditions in other countries on the Continent: Trigonometric developments were brought together and expounded more or less completely, to make them accessible to a wider circle of readers; but in this the abbreviated notations and the beginnings in the use of formulas, which we have repeatedly encountered in England, received little attention and application."[1]

524. *The eighteenth century.*—During the eighteenth century the use of abbreviations for the trigonometric lines became general on the European Continent, but the symbolism differed from that most popular in England in not being so intensely specialized as to be represented by a single letter. As a rule, the Continent used three letters (as in "sin.," "cos.," "tan.," etc.) in their formulae, while the English, between 1700 and about 1750 or 1760, used only one or two letters (as in *s*, *cs*, *t*). There are of course eighteenth-century authors in England, who use the three-letter abbreviations, just as there are writers on the mainland who employ only one or two letters. But, in general, there existed the difference which we have mentioned. Then again, as a rule, the continental writers used dots after the abbreviations, while the English did not. The trend of eighteenth-century practice is established in the following tables.

But we first give symbols proposed by Johann III Bernoulli (Jean Bernoulli, 1744–1807)[2], which do not lend themselves to tabular exhibition. In an effort to produce a symbolic trigonometry for writing succinctly the fundamental theorems, he introduced for right spherical triangles the symbols: $\lrcorner$ *l'angle droit,* $\smile$ *l'hypothénuse,* $\wedge$ *un angle,* $\frown$ *un côté,* $\bar{\wedge}$ *l'autre angle,* $\frown$ *l'autre côté,* $\ltimes$ *l'angle adjacent,* $\wedge$ *l'angle opposé,* $\prec$ *le côté adjacent,* $\succ$ *le côté opposé.* For oblique spherical triangles he introduced the symbols: $\triangle$ *un triangle sphérique rectangle,* $\triangle$ *un triangle sphérique obliquangle,* $\triangle$ *tout triangle sphérique rectangle,* $\triangle$ *tout triangle sphérique obliquangle,* $\sqcap$ *de même espece,* $\dashv$ *de differente espece,* $\vee$ *aigu,* $\vee$ *obtus,* SS *segmens,* $\wedge\!\smile\!\wedge$ *angles à la base,* $\wedge$ *angles au sommet.* He writes the formulae: Dans $\triangle$ le R:ſin. $\smile$ :: ſin. $\wedge$:ſin. $\succ$; Dans $\triangle$ les ſinus des SS :: les cotgts des $\wedge\!\smile\!\wedge$.

[1] A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Vol. II (Leipzig, 1903), p. 50.

[2] Jean Bernoulli, *Recueil pour les Astronomes*, Tome III (Berlin, 1776), p. 133, 158, and Planche V. For this reference I am indebted to R. C. Archibald, of Brown University.

525. *Trigonometric symbols of the eighteenth century.*—

No.	Date	Author	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant	Versed Sine
1....	1704	W. Leybourn	s	cs	t	ct	....	....	....
2....	1706	W. Jones	s	s	t	t	f	$\int$	ν
3....	1753	W. Jones	s	s̀	t	t̀	f	f	ν
4....	1714	J. Wilson	S, s	Σ, σ	T, t	T	....	....	VS
5....	1714	E. Wells	s	σ	t	....	....	....	....
6....	1720	J. Kresa	S.	S. 2.	T.	T. 2.	Sec.	S. 2.	....
7....	1726	J. Keill	S.	Cos.	T.	Cot.	....	....	V.S.
8....	1727	Ph. Ronayne .	S	Σ	T	τ	$\int$	σ	v
9....	1727	F. C. Maier	S, s	C, c	T, t	....	....	....	....
10....	1729	L. Euler	$\int$	co$\int$, cos	....	....	....	....	....
11....	1730	J. Ward	S	Σ	T	Tc	....	....	....
12....	1737	Th. Simpson	Sine	Co-sine	Tangent	Co-tangent	Secant	....	....
13....	1750	Th. Simpson	Sin.	Co-$\int$.	Tang.	Cot., Co-t.	....	Co-seca.	....
14....	1748	L. Euler	sin.	cos.	tang.	cot.	sec.	cosec.	....
15....	1753	L. Euler	sin, sn	cos, cs	tang, tag, tng, tg	....	....	....	....
16....	1755	C. E. L. Camus	S.	co–S.	T	co-T	Séc.	co-Séc.	....
17....	1758	A. G. Kästner	sin	cos, Cos, cosin	tang, Tang	cot	sec, Sec	cosec	....
18....	{1754, 1763}	D'Alembert	Sin., sin., sin	Cos., cos., cos	tang.	cot.	....	....	....
19....	1762	E. Waring	s, S, σ, sin.	....	Tan.	....	Sec	Sec Com	....
20....	1765	A. R. Mauduit	s	c	t	T	S	S	ν
21....	1767	J. A. Segner	sin.	cos.	tan.	cot.	....	....	....
22....	1768	D. F. Rivard	sin.	....	tang.	....	séc.	....	....
23....	1770	P. Steenstra	S.	Cos.	T.	CoT., Cot.	....	....	....
24....	1770	S. Klügel .	sin	cos	tang	cot	sec	cosec	....
25....	1772	C. Scherffer	sin.	cos.	tang.	cot.	sec.	....	....
26....	1772	G. de Koudon	sin.	cos.	tang.	....	sec.	coséc.	sinv.
27....	1772	O. Gherli	Sen	Cos.	Tang.	Cot.	Sec.	Cosec.	....
28....	1774	J. Lagrange	sin	cos	tang, tang.	....	....	....	....

No.	Date	Author	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant	Versed Sine
29	1774	Sauri	sin.	co-sin.	tang.	co-tang.	séc.	co-séc.	....
30	1778	L. Bertrand	sin.	cos.	tang.	cot.	sec.	cosec.	....
31	1782	P. Frisius	sin.	cos.	tang.	cot.	sec.	cosec.	....
32	1782	P. Ferroni	Sin.	Cosin.	Tang.	Cotang.	Sec.	Cosec.	....
33	1784	G. Vega	sin	cos	tang	cot	sec	cosec	sinvers
34	1786	J. P. de Gua	Σ, Π, Γ σ, π, γ	S, P, G s, p, g	⅃Σ, ⅃Π, ⅃Γ ⅃σ, ⅃π, ⅃γ	⅃S, ⅃P, ⅃G ⅃s, ⅃p, ⅃g	$\Gamma\Sigma, \Gamma\Pi, \Gamma\Gamma$ $\Gamma\sigma, \Gamma\pi, \Gamma\gamma$	$\Gamma S, \Gamma P, \Gamma G$ $\Gamma s, \Gamma p, \Gamma g$	....
35	1786	J. B. J. Delambre	sin.	cos.	tang.	cot.	séc.	coséc.	....
36	1786	A. Cagnoli	sin.	cos.	tang.	cot.	séc.	coséc.	....
37	1787	D. Bernoulli	sin.	cos.	tang.	....	....	....	....
38	1787	J. J. Ebert	Sin.	Cos.	Tang., T.	Cot.	....	....	....
39	1788	B. Bails	sen.	cos., cosen.	tang.	cotang.	secante	cosec.	....
40	1790	J. A. Da Cunha	sen	cos	tang	cot.	sec	cosec	....
41	1794	P. Paoli	sen.	cos.	tang.	cot.	sec.	cosec.	....
42	1795	S. l'Huilier	sin.	cos.	tang.	cot.	sec.	....	....
43	1795	Ch. Hutton	sin., s.	cosin., s′	tan., t.	cotan., t′	....	....	....
44	1797	E. Bézout	sin.	cos.	tang.	cot.	séc.	....	....
45	1799	von Metzburg	Sin., S.	Cos.	T.	Cot.	Sec.	Cosec.	Sin. vers.
46	1800	J. F. Lorenz	sin	cos	tang	cot	....	....	....

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3. William Jones, *Philosophical Transactions* (London, 1753), p. 560.
4. John Wilson, *Trigonometry* (Edinburgh, 1714), p. 41, 43.
5. Edward Wells, *The Young Gentleman's Trigonometry* (London, 1714), p. 15, 17.
6. Jakob Kresa, *Analysis speciosa Trigonometriae* (Pragae, 1720), p. 288.
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12. Thomas Simpson, *Treatise of Fluxions* (London, 1737), p. 176, 192.
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14. L. Euler, *Introductio in analysin infinitorum* (Berlin, 1748), Vol. I, p. 93, 94, 103, 104.
15. Leonard Euler, *Histoire de l'Académie de Berlin* (Berlin, 1753), p. 223–57, 258–93. Euler exerted himself to arithmetize trigonometry and to use sin x, cos x, and tan x as functions of x. His efforts along this line are especially displayed in his article, "Svbsidium calcvli sinvvm," of 1751, where, conscious of the importance of his innovation, he states: "Simili autem modo mihi equidem angulorum sinus tangentesque primus in calculum ita transtulisse videor, ut instar reliquarum quantitatum tractari, cunctaeque operationes sine vllo impedimento peragi possent. . . . . Tamen haec ipsa notandi ratio post modum vniversae analysi tanta attulit adiumenta, vt nouum fere campum patefecisse videatur, in quo Geometrae non sine notabili elaborauerint fructu. Ac si quidem ipsus Analysis praestantiam spectamus, eam praecipue soli idoneo quantitates signis denotandi modo tribuendam esse deprehendimus, quo minus erit mirandum, si commoda sinuum in algorithmum introductio tantum lucri attulerit." ("Similarly I believe that I have first introduced the sines and cosines of angles into the calculus, so that they can be treated like other quantities and can be dealt with in extended operations without any hindrance . . . . so that this very mode of designation brings to general analysis great aid, that almost a new field is thereby seen to be opened up, in which geometers have worked not without great results. And if we behold the excellence of analysis we recognize that it must be attributed to the mode of denoting quantities by signs, hence it is less to be marveled if the skilful introduction of the sine into the algorithm has brought such great advantage.")
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32. Pietro Ferroni, *Magnitudinum exponentialium logarithmorum nova methodo pertractata* (Florentiae, 1782), p. 537.
33. Georg Vega, *Vorlesungen über die Mathematik* (Wien), Band II (1784), p. 180–84.
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526. Trigonometric symbols of the nineteenth century.—

No.	Date	Author	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
1...	1803	S. F. Lacroix	sin	cos	tang	cot	séc	coséc
2...	1811	S. D. Poisson	sin.	cos.	tan.	....	....	....
3...	1812	J. Cole	sin.	cos.	tan.	cot. cotan.	sec.	cosec.
4...	1813	A. L. Crelle	sin	cos	tang	cot	sec	cosec
5...	1814	P. Barlow	sin.	cos.	tan.	cot.	sec.	cosec.
6...	1814	C. Kramp	Sin.	Cos.	Tang.	Cot.	....	....
7...	1817	A. M. Legendre	sin	cos	tang	cot	séc	coséc
8...	1823	J. Mitchell	sin.	cos.	tan.	cotan.	sec.	cosec.
9...	1823	G. U. A. Vieth	sin.	cos.	tang.	cot.	sec.	....
10...	1826	N. H. Abel......	sin	cos	tang	....	....	....
11...	1827	J. Steiner	....	....	tg	....	....	....
12...	1827	C. G. J. Jacobi	sin	cos	tg, tang.	cotg	....	....
13...	1829	M. Ohm	Sin	Cos	Tg	Cotg	Sec	Cosec
13*a*..	1830	W. Bolyai	ⓢx	Ⓢx	ⓣx	Ⓣx	Ⓢx	Ⓢx
14...	1836	J. Day	sin	cos	tan	cot	sec	cosec
15...	1840	A. Cauchy	sin	cos	tang	....	....	....
16...	1861	B. Peirce	sin.	cos.	tang.	cotan.	sec.	cosec.
17...	1862	E. Loomis	sin.	cos.	tang.	cot.	sec.	cosec.
18...	1866	F. M. Pires	Sen	Cos	tg	Cot	....	....
19...	1875	J. Cortazar	sen	cos	tg	cot	....	....
20...	1875	I. Todhunter	sin	cos	tan	cot	sec	cosec
21...	1880	J. A. Serret	sin	cos	tang	....	....	....
22...	1881	Oliver, Wait, and Jones	sin	cos	tan	cot	sec	csc
23...	1886	A. Schönflies	....	....	tg	ctg	....	....
24...	1890	W. E. Byerly	sin	cos	tan	ctn	sec	csc
25...	1893	O. Stolz	sin	cos	tan	cot	sec	cosec
26...	1894	E. A. Bowser	sin	cos	tan	cot	sec	cosec
27...	1895	C. L. Dodgson			....	....	....	....
28...	1897	G. A. Wentworth	sin	cos	tan	cot	sec	csc
29...	1903	G. Peano	sin, s	cos , c	tng, t	/tng	/cos	/sin
30...	1903	Weber, Wellstein	sin	cos	tg	cotg	sec	cosec
31...	1911	E. W. Hobson	sin	cos	tan	cot	sec	cosec
32...	1913	Kenyon, Ingold	sin	cos	tan	ctn	sec	csc
33...	1917	L. O. de Toledo	sen	cos	tg	ctg	sec	cosec
34...	1911	A. Pringsheim, J. Molk	sin	cos	tg	cot	séc	coséc
35...	1921	H. Rothe	sin	cos	tg	cot	sec	cosec
36...	1921	G. Scheffers......	sin	cos	tg	ctg	....	....

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36. Georg Scheffers, *Lehrbuch der Mathematik* (Berlin und Leipzig), 5. Aufl. (1921), p. 498.

527. *Less common trigonometric functions.*—The frequent occurrence in practice of certain simple trigonometric expressions has led to suggestions of other functions. Thus, in navigation $\frac{1}{2}(1-\cos A)$ has led to symbolisms such as[1] "$\frac{1}{2}$ sinver A" and "hav A" (i.e., haversine or half the versed sine of A). Another symbol "covers A"[2] or "cvs A"[3] stands for $1-\sin A$. W. Bolyai[4] marked the versed cosine of x by

[1] A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie,* 2. Teil (Leipzig, 1903), p. 231; P. R. Rider and A. Davis, *Plane Trigonometry* (New York, 1923), p. 42. The haversine function first appears in the tables of logarithmic versines of José de Mendoza y Rios (Madrid, 1801, also 1805, 1809), and later in a treatise on navigation of James Inman (1821). See J. D. White in *Nautical Magazine* (February and July, 1926).

[2] William Jones, op. cit., in 1706, represented the coversed sine by v, and A. R. Mauduit, *op. cit.*, in 1765, by u. The designation "covers A" is found, for example, in G. A. Wentworth, *Trigonometry* (2d ed.; Boston, 1903), p. 5; H. H. Ludlow, *Trigonometry* (3d ed., 1891), p. 33; A. M. Kenyon and L. Ingold, *Trigonometry* (New York, 1913), p. 8, 9.

[3] F. Anderegg and E. D. Rowe, *Trigonometry* (Boston, 1896), p. 10.

[4] Wolfgang Bolyai, *Az arithmetica eleje* (Maros-Vásárhelyt, 1830). See B. Boncompagni, *Bullettino,* Vol. I (1868), p. 287.

$\mathfrak{S}x$, and the versed sine of x by $\odot$. A third, "exsec A," i.e., "external secant of A," signifies[1] sec $A-1$.

528. *Quaternion trigonometry.*—I. Stringham[2] outlined a quaternion trigonometry by defining the cosine and sine of the angle between the vectors α and β in a right triangle whose sides are the vectors α, β, δ, in this manner, $\text{cq}\,\frac{\beta}{\alpha}=\frac{\alpha}{\beta}$, $\text{Sq}\,\frac{\beta}{\alpha}=\frac{\delta}{\beta}$, where "cq" signifies "quaternion cosine" and "sq" signifies "quaternion sine." He obtains $\text{sq}\,\frac{\beta}{\alpha}+\text{cq}\,\frac{\beta}{\alpha}=1$, which yields in scalar trigonometry $\text{T}^2\text{sq}\,\frac{\beta}{\alpha}+\text{T}^2\,\text{cq}\,\frac{\beta}{\alpha}=1$, "T" meaning "tensor."

529. *Hyperbolic functions* were first introduced in 1757 by Vicenzo Riccati[3] who used the notation Shx, Chx for hyperbolic sine and cosine. He writes our $\cosh^2 x-\sinh^2 x=1$ in this manner, "$\overline{\text{Ch}.\mu}^2-\overline{\text{Sh}.\ \mu}^2=r^2$," where μ is the arc and r the *sinus totus*. Hyperbolic functions were further developed in 1768 by J. H. Lambert[4] who writes "sin h$(k-y)$" and "cos hk" for the hyperbolic sine and cosine. This use of "h" after the ordinary trigonometric contractions has retained its place in many books to the present time. Seventeen years after Riccati, L'Abbé Saurin[5] represented the *sinus hyperbolique* by "s.h.," the *co-sinus hyperbolique* by "c.h.," the *tangente hyperbolique* by "t.h.," the *co-tangente hyperbolique* by "cot.h.," and wrote

$$\text{"c.h. } n.\ x=\frac{(\text{c.h. } x+\text{s.h. } x)^n+(\text{c.h. } x-\text{s.h. } x)^n}{2.r^{n-1}}\text{."}$$

Somewhat later Frullani[6] writes

$$\text{"}\int_0^\infty \text{Cos h sin h } x \text{ dh}=\pi\text{."}$$

[1] R. G. Hudson and J. Lipka, *A Manual of Mathematics* (New York, 1917), p. 68; A. M. Kenyon and L. Ingold, *Trigonometry* (1913), p. 5.

[2] I. Stringham in *Johns Hopkins University Circulars*, I (1880), p. 35. See von Braunmühl, *op. cit.*, Vol. II, p. 246.

[3] *Institutiones analyticae a Vincentio Riccato . . . . et Hieronymo Saladino* . . . . Tomus Secundus (Bologna, 1767), p. 152.

[4] *Histoire de l'académie de Berlin* (1768), Vol. XXIV, p. 327; reference taken from S. Günther, *Lehre von den Hyperbelfunktionen* (Halle a/S., 1881).

[5] L'Abbé Sauri, *Cours complet de Mathématiques*, Tome IV (Paris, 1774), p. 222, 223.

[6] Frullani in *Mem. di mat. e di fis. della società italiana delle scienze*, Tomo XX, p. 66. See S. Günther, *op. cit.*, p. 36.

J. Houël[1] in 1864 uses "Sh," "Ch," "Th," and introduces "Amh u" to represent what he called the "hyperbolic amplitude of u" in analogy with the amplitude of an elliptic function. J. A. Serret[2] gives in 1857 the symbolism "cos. h. x," "sin. h. x," "tang. h. x."

For hyperbolic functions, E. W. Hobson writes $\cosh u = \cos \iota u$, $\sin hu = -\iota \sin \iota u$, $\tan \text{h}\, u = -\iota \tan \iota u$, and similarly, "coth," "sech," "cosech."

A new notation as well as a new nomenclature was introduced in 1829 by C. Gudermann[3] in an article on "Potential-Funktion." He uses a capital initial letter in the ordinary trigonometric function for the designation of the hyperbolic functions. Moreover, the type used is German. Regarding this choice he himself says:[4] "Finally the choice of the initial syllables printed with German type: $\mathfrak{Cos}$, $\mathfrak{Sin}$, $\mathfrak{Tang}$, $\mathfrak{Cot}$, $\mathfrak{Arc.}\ \mathfrak{Sin}$, $\mathfrak{Arc.}\ \mathfrak{Cos}$, $\mathfrak{Arc.}\ \mathfrak{Tang}$, $\mathfrak{Arc.}\ \mathfrak{Cot}$, and the sign $\mathfrak{L}$ may not meet with favor outside of Germany. But one can take instead the initial syllables written in Latin capitals and for $\mathfrak{L}$ take L and thus by the use of capital letters express the difference from the cyclic functions." Also (p. 290): "To carry out a consideration relating to the construction of the tables, I set up the formula $\mathfrak{Cos}\ \phi .\ \cos k = 1$ which expresses a relation between the two arcs ϕ and k which I designate by $\phi = \mathfrak{L}k$ and conversely by $k = l\phi$. I take $\mathfrak{L}k$ as the Längen-Zahl for the arc k, $\mathrm{l}\phi$ as the Longitudinal-Zahl for the arc ϕ."

Gudermann's notation has been widely used in Germany. It was used in expository articles by Wilhelm Matzka,[5] J. A. Grunert,[6] Siegmund Günther,[7] and L. A. Sohncke,[8] and as late as 1908 by H. E. Timerding[9] who writes in German type $\mathfrak{Sin}$, $\mathfrak{Cos}$, $\mathfrak{Tang}$, $\mathfrak{Sec}$, as does

[1] J. Hoüel in *Nouvelles Annales de Mathématiques* (2d ser.), Vol. III (1864), p. 426, 432.

[2] J. A. Serret, *Traité de Trigonométrie* (2d ed.; Paris, 1857), p. 217.

[3] Gudermann in *Crelle's Journal*, Vol. IV (1829), p. 288.

[4] C. Gudermann, "Ueber die Potenzial-Functionen," in *op. cit.*, Vol. IV (1829), p. 295.

[5] W. Matzka in *Archiv der Math. u. Physik*, Vol. XXXVII (1861), p. 408.

[6] J. A. Grunert, *op. cit.*, Vol. XXXVIII (1862), p. 49.

[7] S. Günther, *Lehre von Hyperbelfunktionen* (Halle a/S., 1881).

[8] L. A. Sohncke, *Sammlung von Aufgaben aus der Differen. u. Integ. Rechnung* (Halle, 1885), p. 9.

[9] H. E. Timerding, *Geometrie der Kräfte* (Leipzig, 1908), p. 330. Consult this for quaternion notations (p. 18).

also H. Rothe[1] in 1921; and in 1912 by Hans von Mangoldt[2] who, however, uses Latin type. On the other hand, J. Frischauf[3] prefers small letters of German black-faced type.

530. The German notation and nomenclature made itself felt in America. Benjamin Peirce[4] in 1846 called the hyperbolic functions "potential functions" and designated them by the symbols for the ordinary trigonometric functions, but with initial capital letters; for example, Sin. $B=\frac{1}{2}(e^B-e^{-B})$. It is found also in articles that appeared in J. D. Runkle's *Mathematical Monthly*, for instance, Volume I, page 11.

However, the Gudermannian notation finally gave way at Harvard to that prevalent in France; in James Mills Peirce's *Mathematical Tables* (Boston, 1881), one finds "Sh u," "Ch u."

In England, W. K. Clifford[5] proposed for the hyperbolic sine, cosine, and tangent the signs "hs," "hc," and "ht," or else "sh," "ch," "th." In the United States, Clifford's first suggestion was adopted by W. B. Smith.[6] Prefixing the syllable "hy" was suggested by G. M. Minchin[7] in 1902. He would write "hy sin x," "hy tg x" in order to facilitate enunciation. This suggestion was adopted by D. A. Murray[8] in his *Calculus* of 1908.

531. *Parabolic functions.*—James Booth, in his *Theory of Elliptic Integrals* (1851), and in an article in the *Philosophical Transactions* for the year 1852,[9] develops the trigonometry of the parabola. For the parabola we have $\tan\omega=\tan\phi\sec\chi+\tan\chi\sec\phi$, ω, ϕ, χ being parabolic arcs. If we make the imaginary transformations $\tan\omega=i\sin\omega'$, $\tan\phi=i\sin\phi'$, $\tan\chi=i\sin\chi'$, $\sec\phi=\cos\phi'$, $\sec\chi=\cos\chi'$, then the foregoing formula becomes $\sin\omega'=\sin\phi'\cos\chi'+\sin\chi'\cos\phi'$, which is the ordinary expression for the sine of the sum of two circular arcs. In the trigonometry of the circle $\omega=\phi+\chi$; in the trigonometry of the parabola ω is such a function of ϕ and χ as will render

[1] Hermann Rothe, *Vorlesungen über höhere Mathematik* (Wien, 1921), p. 107.

[2] Hans von Mangoldt, *Einführung in die höhere Mathematik*, Vol. II (Leipzig, 1912), p. 518; Vol. III (1914), p. 154, 155.

[3] Johannes Frischauf, *Absolute Geometrie* (Leipzig, 1876), p. 54.

[4] Benjamin Peirce, *Curves, Functions, and Forces*, Vol. II (1846), p. 27, 28.

[5] W. K. Clifford, *Elements of Dynamic*, Part I (London, 1878), p. 89.

[6] W. B. Smith, *Infinitesimal Analysis*, Vol. I (New York, 1897), p. 87, 88.

[7] G. M. Minchin in *Nature*, Vol. LXV (London, 1902), p. 531.

[8] D. A. Murray, *Differential and Integral Calculus* (London, 1908), p. 414.

[9] James Booth, "On the Geometrical Properties of Elliptic Integrals," *Philosophical Transactions* (London, 1852), p. 385.

tan $[(\phi, \chi)]=\tan\phi \sec\chi+\tan\chi \sec\phi$. Let this function (ϕ, χ) be designated $\phi\perp\chi$, so that $\tan(\phi\perp\chi)=\tan\phi\sec\chi+\tan\chi\sec\phi$; let it be designated $(\phi\top\chi)$ when $\tan(\phi\top\chi)=\tan\phi\sec\chi-\tan\chi\sec\phi$. Thus when $\tan\phi$ is changed into $i \sin\phi$, $\sec\phi$ into $\cos\phi$, and $\cot\phi$ into $-i \operatorname{cosec}\phi$, $\perp$ must be changed into $+$, $\top$, into $-$.

532. *Inverse trigonometric functions.*—Daniel Bernoulli was the first to use a symbolism for inverse trigonometric functions. In 1729 he used "A S." to represent "arcsine."[1] Euler[2] in 1736 introduced "A t" for "arctangent," in the definition: "expressio A t nobis denotet arcum circuli, cuius tangens est t existente radio=1." Later, in the same publication, he expressed[3] the arcsine simply by "A": "arcus cuius sinus est $\frac{x}{a}$ existente toto sinu=1 notetur per A $\frac{x}{a}$." To remove the ambiguity of using "A" for two different arc functions, he introduced[4] the sign "A t" for "arctangent," saying, "At.$\frac{q}{B}$ est arcus circuli cuius tangens est $\frac{q}{B}$ existente sinu toto=1." He used the sign "A t" in 1736 also in another place.[5] In 1737 Euler[6] put down "A sin $\frac{b}{c}$" and explained this as meaning the arc of a unit-circle whose sine is $\frac{b}{c}$. In 1744 he[7] uses "A $\overline{\text{tagt}}$" for *arcus, cujus tangens*$=t$. Lambert[8] fourteen years later says, "erit $\frac{x}{m}=\frac{x}{o}$ arcus sinui b respondens." Carl Scherffer[9] in Vienna gives "arc. tang." or "arc. tangent"; J. Lagrange[10] in the same year writes "arc. sin $\frac{1}{1+a}$." In

[1] Daniel Bernoulli in *Comment. acad. sc. Petrop.*, Vol. II (1727; printed 1728), p. 304–42. Taken from G. Eneström, *Bibliotheca mathematica* (3d ser)., Vol. XIV, p. 78. See also Vol. VI (1905), p. 319–21.

[2] L. Euler, *Mechanica sive motus scientia* (Petropoli, 1736), Vol. I, p. 184–86.

[3] L. Euler, *op. cit.*, Vol. II, p. 138.

[4] L. Euler, *op. cit.*, Vol. II, p. 303.

[5] L. Euler, *Comment. acad. sc. Petrop.*, Vol. VIII (1736; printed in 1741), p. 84, 85.

[6] L. Euler in *Commentarii academiae Petropolitanae ad annum 1737*, Vol. IX, p. 209; M. Cantor, *op. cit.*, Vol. III (2d ed.), p. 560.

[7] *Nova acta eruditorum* (1744), p. 325; Cantor, *op. cit.*, Vol. III (2d ed.), p. 560.

[8] *Acta Helvetica Physico-Math.-Botan.*, Vol. III (Basel, 1758), p. 141.

[9] Carolo Scherffer, *Institutionum analyticarum pars secunda* (Vienna, 1772), p. 144, 200.

[10] J. Lagrange in *Nouveaux mémoires de l'académie r. d. sciences et belles-lettres*, année 1772 (Berlin, 1774), p. 277.

1776 Lambert[1] uses "arc. sin." The notation was adopted by French writers. Marquis de Condorcet[2] wrote arc $(\cos.=(1+P)^{-\frac{1}{2}})$ for our arc $\cos (1+P)^{-\frac{1}{2}}$. Bérard gave "Arc. $\left(\text{Tang.}=\frac{dy}{dx}\right)$." In Italy, Pietro Paoli[3] adopts "Arc. sen." for "arcsin."

533. *John Herschel's notation for inverse functions*, $\sin^{-1} x$, $\tan^{-1} x$, etc., was published by him in the *Philosophical Transactions of London*, for the year 1813. He says (p. 10): "This notation $\cos.^{-1} e$ must not be understood to signify $\frac{1}{\cos. e}$, but what is usually written thus, arc $(\cos.=e)$." He admits that some authors use $\cos.^m A$ for $(\cos. A)^m$, but he justifies his own notation by pointing out that since d^2x, Δ^3x, Σ^2x mean ddx, $\Delta\Delta\Delta x$, $\Sigma\Sigma x$, we ought to write $\sin.^2 x$ for $\sin. \sin. x$, $\log^3 x$ for $\log. \log. \log. x$. Just as we write $d^{-n} V=\int^n V$, we may write similarly $\sin.^{-1} x=\text{arc} (\sin.=x)$, $\log.^{-1} x.=c^x$. Some years later Herschel explained that in 1813 he used $f^n(x)$, $f^{-n}(x)$, $\sin.^{-1} x$, etc., "as he then supposed for the first time. The work of a German Analyst, Burmann, has, however, within these few months come to his knowledge, in which the same is explained at a considerably earlier date. He [Burmann], however, does not seem to have noticed the convenience of applying this idea to the inverse functions $\tan^{-1}$, etc., nor does he appear at all aware of the inverse calculus of functions to which it gives rise." Herschel adds, "The symmetry of this notation and above all the new and most extensive views it opens of the nature of analytical operations seem to authorize its universal adoption."[4] Thus was initiated a notation which has kept its place in many English and American books to the present time.

534. *Martin Ohm's notation for inverse functions.*—In Germany, Ohm introduced a new notation for the inverse trigonometric functions which found no favor. He says: "What is here designated by $\frac{1}{\text{Sin}} y$, $\frac{1}{\cos} y$, etc., is usually indicated by arc. Sin y, arc. Cos y, etc., also by ang. $(\text{Sin}=y)$, ang. $(\text{Cos}=y)$, etc. Here this new notation is recommended because it admits a much more convenient mechanism

[1] Lambert in *Nouveaux mémoires de l'académie r. d. sciences et belles-lettres*, année 1776 (Berlin, 1779), p. 12.

[2] N. C. de Condorcet, *Histoire de l'académie r. d. sciences*, année 1769 (Paris, 1772), p. 255.

[3] Pietro Paoli, *Elementi d'algebra* (Pisa, 1794), Vol. II, p. 21.

[4] John F. W. Herschel, *Collection of Examples on Calculus of Finite Differences*, Cambridge, 1820, p. 5, 6.

for the calculus, inasmuch as, analogous to the usual forms of algebra, there follows from $\frac{1}{\mathrm{Tg}}\, y=x$ immediately $y=\mathrm{Tg}x$; from $\frac{1}{\mathrm{Cos}}\, z=u$, immediately $z=\mathrm{Cos}\, u$; etc."[1]

535. *Persistence of rival notations for inverse functions.*—The eighteenth-century notations, involving the word "arc" or its equivalent, were only mild contractions of the words which defined the meanings to be conveyed to the reader, and they maintained their place in most works of Continental Europe during the nineteenth century. C. Gudermann[2] in 1829 used the notation "arc ($\mathrm{tang}=z$)," etc. In France, J. Houël employed the easier form "arcsin." The only marked inroad of Herschel's notation upon the Continent is found with G. Peano who in 1893[3] adopts the Eulerian $\overline{\cos}\, x$, for arc cos x, and later[4] writes $\sin^{-1}$, $\cos^{-1}$, tang^{-1}.

In Mexico the continental and English notations clashed. F. D. Covarrubias[5] writes "arco ($\sec=x$)" and lets $\cos^{-1} x$ stand for $\frac{1}{\cos x}$. On the other hand, Manuel Torres Torija[6] uses "$x=\tan^{-1} y$" and says that it means the same as "$x=\mathrm{arc}\ (\tan=y)$."

In England, Herschel's notation gained ground rapidly, but not instantaneously. We find, for instance, in 1838, "arc $\left(\tan.\ \frac{x}{c}\right)$" in a treatise by John West.[7]

In the United States the British influence greatly predominated during the nineteenth century. A rare exception is the occurrence of the continental symbols "arc tan," "arc csc," in a book of E. P. Seaver.[8] The use of Herschel's notation underwent a slight change in

[1] M. Ohm, *System der Mathematik*, Vol. II (Berlin, 1829), p. 372.

[2] C. Gudermann, *Journal für r. u. a. Mathematik*, Vol. IV (1829), p. 287.

[3] G. Peano, *Lezioni di analisi infinitesimale*, Vol. I (Turin, 1893), p. 43.

[4] G. Peano, *Formulaire mathématique, Edition de l'an 1902–'03* (Turin, 1903), p. 228, 229.

[5] F. D. Covarrubias, *Elementos de análisis trascendente ó cálculo infinitessimal* (2d ed.; Mexico, 1890), p. 48, 49.

[6] Manuel Torres Torija, *Nociones de Álgebra Superior y elementos fundamentales de cálculo differencial é integral* (México, 1894), p. 181.

[7] Rev. John West, *Mathematical Treatises* (ed. John Leslie; Edinburgh, 1838), p. 237.

[8] Edwin P. Seaver, *Formulas of Plane and Spherical Trigonometry* (Boston and Cambridge, 1871), p. 43.

Benjamin Peirce's books, to remove the chief objection to them. Peirce wrote:

$$\text{“}\cos^{[-1]} x,\text{” “}\tan^{[-1]} x\,.\text{”}$$[1]

In the present century the continental notation of Europe has found entrance in the United States through several texts, as, for instance, those of Love,[2] Wilczynski and Slaught,[3] Kenyon and Ingold.[4]

536. *Inverse hyperbolic functions.*—Houël[5] marked the inverse hyperbolic sine, cosine, and tangent thus: "Arg Sh," "Arg Ch," "Arg Th." Similar notations have been used by other writers. The arcsin tanh u is called by Cayley the Gudermannian of u and is written[6] "gd u." When z is a complex variable, the symbolism "Arg sh z" was used by A. Pringsheim, G. Faber, and J. Molk[7] in their joint article in the *Encyclopédie* to indicate the infinitely fold function $\text{Arg sh}\, z = \frac{\text{Arc sin}\, iz}{i}$, but the authors point out that in one respect the notation is improper, because the function which is the inverse of "sh$z$" is not defined by the aid of the arc of the curve.

537. *Powers of trigonometric functions.*—Three principal notations have been used to denote, say, the square of sin x, namely, $(\sin x)^2$, $\sin x^2$, $\sin^2 x$. The prevailing notation at present is $\sin^2 x$, though the first is least likely to be misinterpreted. In the case of $\sin^2 x$ two interpretations suggest themselves: first, $\sin x \cdot \sin x$; second,[8] $\sin(\sin x)$. As functions of the last type do not ordinarily present themselves, the danger of misinterpretation is very much less than in case of $\log^2 x$, where $\log x \cdot \log x$ and $\log(\log x)$ are of frequent occurrence in analysis. In his *Introductio in analysin* (1748), Euler[9] writes $(\cos . z)^n$, but in an article of 1754 he adopts[10] $\sin \cdot \psi^3$ for $(\sin \psi)^3$ and writes the

[1] B. Peirce, *Curves, Functions and Forces*, Vol. I (new ed.; Boston, 1852), p. 203.

[2] C. E. Love, *Differential and Integral Calculus* (New York, 1910), p. 132.

[3] E. J. Wilczynski and H. E. Slaught, *Plane Trigonometry* (1914), p. 217.

[4] A. M. Kenyon and L. Ingold, *Trigonometry* (1913), p. 102, 105.

[5] J. Houël, *Cours de calcul infinitésimal*, Vol. I (Paris, 1878), p. 199.

[6] G. Chrystal, *Algebra*, Part II (1889), p. 288.

[7] *Encyclopédie des sciences math.*, Tome II, Vol. II (1911), p. 80.

[8] See G. Peano, *Formulaire mathématique*, Vol. IV (1903), p. 229

[9] L. Euler, *Introductio in analysin infinitorum* (Lausannae, 1748), Vol. I, p. 98, 99.

[10] L. Euler in *Novi Comment. acad. scient. imper. Petropolitanae*, for the years 1754, 1755 (Petropoli, 1760), p. 172.

formula "$4 \sin\cdot\psi^3 = -\sin\cdot 3\,\psi + 3 \sin\cdot\psi$." William Jones in 1710 wrote "$cs^2, \frac{1}{2} < v$" for $\left(\cos\frac{v}{2}\right)^2$, and "$s^2, \frac{1}{2}v$" for $\left(\sin\frac{v}{2}\right)^2$.

The parentheses as in $(\sin x)^n$ were preferred by Karsten,[1] Scherffer,[2] Frisius,[3] Abel (in some passages),[4] Ohm.[5] It passed into disuse during the nineteenth century. Paoli[6] writes "$\overline{\text{sen}.a}^m$." The designation $\sin x^2$ for $(\sin x)^2$ is found in the writings of Lagrange,[7] Lorenz,[8] Lacroix,[9] Vieth,[10] Stolz;[11] it was recommended by Gauss.[12] The notation $\sin^n x$ for $(\sin x)^n$ has been widely used and is now the prevailing one. It is found, for example, in Cagnoli,[13] De Morgan,[14] Serret,[15] Todhunter,[16] Hobson,[17] Toledo,[18] Rothe.[19]

[1] W. J. G. Karsten, *Mathesis theoretica* (Rostock, 1760), p. 511.

[2] Carolo Scherffer, *Institutionum analyticarum pars secunda* (Vienna, 1772), p. 144.

[3] Paulli Frisii, *Operum tomus primus* (Milan, 1782), p. 303.

[4] N. H. Abel in *Crelle's Journal*, Vol. I (Berlin, 1826), p. 318–37; Vol. II (1827), p. 26.

[5] Martin Ohm, *System der Mathematik*, 3. Theil (Berlin, 1829), p. 21.

[6] Pietro Paoli, *Elementi d'algebra*, Vol. II (Pisa, 1794), p. 94, 95, 125, 129.

[7] J. Lagrange, *Méchanique analitique* (Paris, 1788), p. 272, 301.

[8] J. F. Lorenz, *Grundlehren der allgemeinen Grössenberechnung* (Helmstädt, 1800), p. 306, 307.

[9] S. F. Lacroix, *Traité élémentaire de trigonometrie* (Paris, 1803), p. 9–17.

[10] G. N. A. Vieth, *Kurze Anleitung zur Differenzialrechnung* (Leipzig, 1823), p. 40–42.

[11] Otto Stolz, *Grundzüge der Differential und Integralrechnung* (Leipzig, 1893), p. 38–42.

[12] See Grunert, *Archiv der Mathematik*, Vol. XXXVIII, p. 366.

[13] Antonio Cagnoli, *Traité de Trigonométrie* (trad. par Chompré; Paris, 1786), p. 20.

[14] A. de Morgan, *Trigonometry and Double Algebra* (London, 1849), p. 35.

[15] J. A. Serret, *Traité de Trigonométrie* (2d ed.; Paris, 1857), p. 12.

[16] I. Todhunter, *Plane Trigonometry* (6th ed.; London, 1876), p. 19.

[17] E. W. Hobson, *Treatise on Plane Trigonometry* (Cambridge, 1911), p. 19.

[18] L. O. de Toledo, *Tradado de Trigonometria* (tercera ed.; Madrid, 1917), p. 64.

[19] Hermann Rothe, *Vorlesungen über höhere Mathematik* (Wien, 1921), p. 261.

SURVEY OF MATHEMATICAL SYMBOLS USED BY LEIBNIZ

538. *Introduction.*—The leading rôle played by Leibniz in the development of mathematical notations induces us here to present a complete survey of all the mathematical symbols used by him.

Among the seventeenth-century mathematicians active in the development of modern notations a prominent rôle was played by five men—Oughtred, Hérigone, Descartes, Leibniz, and Newton. The scientific standing of these men varied greatly. Three of them—Descartes, Leibniz, and Newton—are generally proclaimed men of genius. The other two—Oughtred and Hérigone—were noted textbook writers. Newton and Descartes did not devote themselves persistently to work on notations. Descartes made only minute changes in the algebraic notations of some of his predecessors, but these changes were generally adopted, along with his great creation in the science of mathematics—his analytic geometry. In the case of Newton, mathematical notation was a minor topic of attention. His notation for fractional and negative exponents was foreshadowed by his predecessors and constituted a fairly obvious extension of the symbols used by Descartes and Wallis. His dot symbolism for fluxions suggested itself to him as a young man, but he made no use of it in his *Analysis* and his *Principia.* Nor did he experiment with it and with rectangles denoting fluents, for the purpose of improving his notation. In an unsigned article, where he[1] speaks of himself in the third person, he declared: "Mr. Newton doth not place his Method in Forms of Symbols, nor confine himself to any particular Sort of Symbols for Fluents and Fluxions."

Unlike Newton and Descartes, Leibniz made a prolonged study of matters of notation; like Oughtred and Hérigone, Leibniz used many new symbols, but there was an important difference of method. Oughtred adopted a large array of symbols in the early editions of his *Clavis,* without having experimented with rival symbols to ascertain their relative merits. For example, his notation for powers of numbers was a modification of that of Vieta; it was strikingly inferior to that of Descartes. Yet Oughtred, in later editions of his *Clavis,* hardly ever modified his own original signs even by a stroke. The same remark applies to Hérigone. The great difference in procedure between Leibniz and his two predecessors was that for about a third of a century Leibniz experimented with different symbols, corre-

[1] *Philosophical Transactions,* Vol. XXIX, for the years 1714–16 (London, 1717), p. 204.

sponded with mathematicians on the subject, and endeavored to ascertain their preferences. It would seem that on this topic he conferred with every mathematician of his acquaintance who was willing to listen sympathetically—with Jakob Bernoulli,[1] Johann Bernoulli,[2] Wallis,[3] Guido Grando,[4] Oldenburgh,[5] Hermann,[6] Huygens,[7] L'Hospital,[8] Tschirnhausen,[9] Zendrini,[10] and Wolf.[11]

539. Perhaps no mathematician has seen more clearly than Leibniz the importance of good notations in mathematics. His symbols for the differential and integral calculus were so well chosen indeed that one is tempted to ask in the words of Goethe's Faust, "War es ein Gott, der diese Zeichen schrieb?" But this excellence was no divine inspiration; it was the result of patient and painstaking procedure. He withheld these symbols from print for about ten years. Before committing himself to print he explained his dx and dy to Oldenburgh[12] and Tschirnhausen.[13] He stressed the importance of a notation which would identify the variable and its differential.[14] The dx indicated the variable x; dy, the variable y; dz, the variable z. This symbolism was immensely superior to the a chosen by Fermat as the small increment of x, and the a and e adopted by Barrows as the increments of x and y. As regards the integral calculus (§ 620, Fig. 124), Johann Bernoulli had been active in this field and was looked upon as the creator of the integral calculus,[15] notwithstanding Leibniz' publication of 1686. At one time Leibniz and Johann Bernoulli discussed in their letters both the name and the principal symbol of the integral calculus. Leibniz favored the name *calculus summatorius* and the long letter $\int$ as the symbol. Bernoulli favored the name *calculus integralis* and the capital letter I as the sign of integration.[16]

[1] *Leibnizens gesammelte Werke. Dritte Folge Mathematik* (*Leibnizens Mathematische Schriften* herausgegeben von C. I. Gerhardt), Vol. III (Halle, 1855), p. 10–110.

[2] *Op. cit.*, Vol. III, p. 133–973.

[3] *Op. cit.*, Vol. IV, p. 5–82.

[4] *Op. cit.*, Vol. IV, p. 209–26.

[5] *Op. cit.*, Vol. I, p. 11–168.

[6] *Op. cit.*, Vol. IV, p. 259–413.

[7] *Op. cit.*, Vol. II, p. 11–208.

[8] *Op. cit.*, Vol. II, p. 216–343.

[9] *Op. cit.*, Vol. IV, p. 429–539.

[10] *Op. cit.*, Vol. IV, p. 230–51.

[11] *Op. cit.*, Vol. VII, p. 14–188 (Wolf).

[12] *Op. cit.*, Vol. I, p. 154.

[13] C. I. Gerhardt, *Der Briefwechsel von G. W. Leibniz mit Mathematikern*, Band I (Berlin, 1899), p. 375.

[14] *Leibnizens Math. Schriften*, Vol. V, p. 231.

[15] *Op. cit.*, Vol. III, p. 115, 116.

[16] *Op. cit.*, Vol. III, p. 115, 168, 170–72, 177, 262, 272, 273; Vol. V, p. 320.

The word "integral" had been used in print first by Jakob Bernoulli,[1] although Johann claimed for himself the introduction of the term.[2] Leibniz and Johann Bernoulli finally reached a happy compromise, adopting Bernoulli's name "integral calculus," and Leibniz' symbol of integration. For many years Leibniz frequently attached a vinculum to his symbol dx for differential and also to his sign of integration. In both cases he later dropped the vinculum, it having been found by experience to be superfluous.

It is not generally known that Leibniz at one time invented a special sign[3] for the differential coefficient, $\frac{dz}{dx}$, namely, ꝺʒ, the letter d being broken. This sign was free from the objection of requiring three terraces of type as is the case in ordinary fractions. He submitted this symbol to Johann Bernoulli who in his reply[4] rather favored its adoption, even to the exclusion of his own capital letter D which he had been using for this purpose. Leibniz himself, doubtless for good reasons, never brought his symbol into print and, in fact, did not again urge its use in his letters. This episode affords a fine example of masterful self-control.

540. In algebra and geometry an improved symbolism on which all could agree was greatly needed. Leibniz engaged in extensive experimentation. At different times he used four different symbols for equality, three for proportion, three for coincidence of geometric figures, two for similarity, four for congruence, five for multiplication, three for division, two for logarithms, and about half a dozen for the aggregation of terms. Besides this he had several different signs for powers and roots, two signs for "greater," and two for "less."

How Leibniz dealt with the problem of notations in his correspondence is illustrated by the following extract from a letter (February 22, 1696) to Johann Bernoulli:[5] "As regards signs, I see it clearly that it is to the interest of the Republic of Letters and especially of students, that learned men should reach agreement on signs. Accordingly I wish to get your opinion, whether you approve of marking by the sign $\int$ the sum, just as the sign d is displayed for differences; also whether you approve of my designation of ratio as if it were a division,

[1] Jakob Bernoulli in *Acta eruditorum* (1690), p. 218; Jakob Bernoulli, *Opera*, Vol. I, p. 423. He says, "Ergo et horum Integralia aequantur."

[2] *Leibnizens Math. Schriften*, Vol. III, p. 115 n., 163, 172.

[3] *Op. cit.*, Vol. III, p. 526.

[4] *Op. cit.*, Vol. III, p. 531.

[5] *Op. cit.*, Vol. III, p. 262.

by two dots, for example, that $a:b$ be the same as $\frac{a}{b}$; it is very easily typed, the spacing of the lines is not disturbed. And, as regards proportion, there are some who exhibit such a relation by $a:b::c:d$; since this really amounts to an equality of quotients, it is sufficient to write, as is my custom, $a:b=c:d$ or $\frac{a}{b}=\frac{c}{d}$. Perhaps it will be well to examine other symbols, concerning which more on another occasion." Bernoulli[1] expresses his assent to these suggestions, although, through force of habit, he continues with his old notations which were somewhat at variance with what Leibniz had proposed. "It is difficult to become adjusted," Bernoulli candidly remarks. Nor is he altogether pleased with Leibniz' colon for division, "since those who are accustomed to the ordinary mark for division $\left[\frac{a}{b}\right]$ can hardly distinguish at a glance between dividend and divisor."

In another letter to Johann Bernoulli, Leibniz says:[2] "In marking operations I diminish the task of type-setting; I use to express aggregation, instead of bars of raised vinculums the direct or inverted commas; thereby the line of type is not broken, nor the spacing disturbed and yet (if I am not deceived) everything is indicated accurately. However, I desire first to ascertain your opinion." In a later communication (July 29, 1698) Leibniz objects to the St. Andrews cross $\times$ which he himself had previously used as a sign of multiplication, because it is easily confused with the letter x. Johann Bernoulli approves of Leibniz' suggestions but frankly adds:[3] "Meanwhile I prefer to follow the accepted practice, rather than begin with the definitions of new signs, which can be done more conveniently when one is writing a whole treatise." Leibniz proceeds[4] next to propose a symbolism for "function," which was forcing its way as a new concept into mathematics. Apparently Leibniz was not satisfied with his symbols for functions, for he did not have them printed, and mathematicians of the eighteenth century invented other designs.

Tschirnhausen expressed the opinion that new terminology and new symbols render the science less comprehensible. He praises Vieta for having used only the letters of the alphabet instead of introducing other characters resembling monsters.[5] But the enthusiasm

[1] *Op. cit.*, Vol. III, p. 273.

[2] *Op. cit.*, Vol. III, p. 276.

[3] *Op. cit.*, Vol. III, p. 531.

[4] *Op. cit.*, Vol. III, p. 537.

[5] C. I. Gerhardt, *Briefwechsel von G. W. Leibniz mit Mathematikern*, Vol. I, p. xvii, 358, 388.

of Leibniz was not so easily crushed; he replies:[1] "I perform the calculus by certain new signs of wonderful convenience, concerning which, when I recently wrote you, you responded that your mode of exposition was the more customary and ordinarily intelligible way and that you avoided, as far as possible, novelty in matters of definition, since it effected indeed nothing but to render the sciences difficult." But the same objection might have been made by the arithmeticians of previous years when the Arabic in the place of the Roman characters were introduced, or by the old algebraists, when Vieta introduced letters in place of numbers. "In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished."

541. During the last fifteen or twenty years of his life, Leibniz reached definite conclusions as to the superiority of certain algebraic symbols over others. In making his decisions he was guided in part by the principle of economy, according to which there should not be unnecessary duplication of symbols. Accordingly, he discarded the four dots of Oughtred in writing proportion. He urged the avoidance of signs so closely resembling each other as to give rise to doubt or confusion. He came to realize the fundamental importance of adopting symbolisms which could be set up in a line like ordinary type,[2] without need of widening the spaces between lines to make room for symbols with sprawling parts. Considerations of this sort led him to discard the vinculum so freely used by him in his earlier practice, not only in the ordinary form for the aggregation of terms in a polynomial, but also in conjunction with the radical sign, the powers of polynomials, the differential dx, and the integral sign. In his later practice the vinculum was displaced by parentheses or by commas or dots. The same principle led him to advocate the printing of fractions in the running text by the use of the colon,[3] a device preferable to the modern solidus. I mention these details to illustrate how painstaking and careful he was. And what was the result of all these efforts, all his experimentation? Simply this, that no other mathematician has advanced as many symbols which have retained their place to the present time as has Leibniz. Of Oughtred's many symbols, the cross in multiplication is the only one which is still universally known and widely used. Of Hérigone's signs all have passed into innocuous

[1] *Op. cit.*, p. 375, 380, 403.

[2] *Leibnizens Mathematische Schriften*, Vol. III, p. 276.

[3] *Op. cit.*, Vol. III, p. 276.

desuetude. Among Leibniz' symbols which at the present time enjoy universal, or well-nigh universal, recognition and wide adoption are his dx, dy, his sign of integration, his colon for division, his dot for multiplication, his geometric signs for similar and congruence, his use of the Recordian sign of equality in writing proportions, his double-suffix notation for determinants.

Leibniz expressed his mathematical creed in a letter to L'Hospital (April 28, 1693) as follows:[1] "One of the secrets of analysis consists in the characteristic, that is, in the art of skilful employment of the available signs, and you will observe, Sir, by the small enclosure [on determinants] that Vieta and Descartes have not known all the mysteries." His ideals for mathematical symbolisms was part of his broader scheme on mathematical logic which has commanded the lively admiration of modern logicians. With exuberant optimism Leibniz prophesied the triumphal success of researches in this field in the famous affirmation:[2] "I dare say that this is the last effort of the human mind, and, when this project shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect not less than the telescope serves to perfect their vision."

542. *Tables of symbols used by Leibniz in his manuscripts and in the papers he published. Abbreviations of titles:*

Ae = *Acta eruditorum.*

B = *Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern,* herausgegeben von C. I. Gerhardt, Band I (Berlin, 1899).

Bm, XIII = *Bibliotheca mathematica* (3d ser.), Vol. XIII (ed. by G. Eneström).

Cat. crit. = *Catalogue critique des Manuscrits de Leibniz,* Fasc. II (Poitiers, 1914–24).

E = *Entdeckung der höheren Analysis,* von C. I. Gerhardt (Halle, 1855).

Ph., VII = C. I. Gerhardt, *Philosophische Schriften von Leibniz,* Vol. VII (Berlin, 1890).

Roman numerals followed by Arabic numerals indicate the volume and page of *Leibnizens gesammelte Werke. Dritte Folge, Mathematik* (*Leibnizens Mathematische Schriften,* herausgegeben von C. I. Gerhardt [1849–63]); thus "V, 196" means Volume V, page 196.

Mah. = Dietrich Mahnke, "Neue Einblicke in die Entdeckungsgeschichte der höheren Analysis," *Abhandlungen der Preuss. Akademie*

[1] *Op. cit.,* Vol. II, p. 240.

[2] Quoted by A. Padoa in *La Logique deductive* (Paris, 1912), p. 21.

der Wissenschaften zu Berlin (Jahrgang 1925), Phys.-math. Kl., No. 1 (Berlin, 1926).

O = *Opuscules et Fragments inédits de Leibniz*, par L. Couturat (Paris, 1903).

P means that the article referred to was printed by Leibniz in the year indicated, or else was printed in the *Commercium epistolicum* of John Collins before the death of Leibniz. The references given in these tables are in most cases only a few out of the total number that might be given.

543. *Differential and derivative.*

Sign	Meaning	Year	Reference
b	Differential, $\xi b = x$	1673	*Cat. crit.*, No. 575; Mah., 45, Table I
ξ	Number of differentials		
a, l	Differentials of x, y	1675	*E*, 123, 125
z	Differential of x	1675	*E*, 132
$\frac{x}{d}$	Differential of x	1675	*E*, 126; *B*, XIV
dx	Differential of x	1675	*E*, 134, 150–55; *B*, XIV; I, 128; II, 60, 108, 118; III, 262, 453; V, 106
∂, ∂x	Differential of x	1684 . . . *P*	V, 220, 257, 310, 315; VII, 222
		1675	*E*, 59, 135; I, 128; II, 44; IV, 479, 506; VII, 355
		1692*P*	V, 256
$\frac{dx}{dy}$	Derivative	1675	*E*, 137, 138; II, 196
$dx:dy$	Derivative	1692 ... 1684*P*	II, 118; III, 288 V, 224
$\frac{dy}{dx}$	Derivative	1676	*E*, 140, 146; *B*, 202, 224, 230; I, 159
ddv	Second differential	1684*P* 1694	V, 221 II, 196; III, 167
$d\overline{dy}$	Second differential	1690	II, 43
$dddy$	Third differential	1697	IV, 25
$d\overline{y^3}$	$d(y^3)$	1676	I, 154; *B*, 229
$\frac{d\overline{x^e}}{dx}$	$= ex^{e-}$	1676	E, 141
$d.x^v$	$= x^v., \frac{v}{x}dx + dv + dv.\log.x$	1695*P*	V, 325*
d^3n	Third differential	1695	III, 167; IV, 25
d^mn	mth differential	1695?	II, 294, III, 167, 221; IV, 25, 211
$d^{\overline{m-1}}n$	$d^{m-1}n$	1695?	II, 294
$d^{-1}x$	$\int x$	1695	II, 274
$d^{1:2}x$	$x . \sqrt[2]{dx:x}$ fract. differential	1695	II, 302; III, 228
$d^{\frac{1}{2}}\,\overline{xy}$	Fractional differential	1695 ...	II, 301; IV, 25
$d^{1:2}\,\overline{xy}$	Fractional differential	1695	II, 301
$dx.dx$	(dx)	?	*E*, 151
$\overline{dx}^2$	$(dx)^2$	1676	*E*, 141
$\mathfrak{d}\mathfrak{z}$	$dz:dx$	1698	III, 526, 531
δm	$\frac{\partial m}{\partial x}$	1694	II, 261
ϑm	$\frac{\partial m}{\partial y}$	1694	II, 261

* Misprint

544. *Signs for integral, sum.*—

Sign	Meaning	Year	Reference
omn. w	Sum of all the w's	1675	*E*, 58, 120, 121; *B*, XII, XIII
omn. $\overline{\text{omn. } w}$	Sum of the sum of all the w's	1675	*E*, 58, 120, 121; *B*, XII, XIII
$\int$	"omn." i.e., sum, integral	1675	*E*, 59, 125, 126, 132–38, 150–55; *B*, XII, XIII; II, 43, 108; III, 104, 116, 167, 262; IV, 506; V, 397
ſ	....	1686 .... *P*	*Ae* (1686), 297–99; (1693), 178
$\int \overline{xl}$	Integral sign and vinculum	1675	*E*, 59, 125, 126, 132–38, 140, 141, 152, 153; *B*, XII, XIII
		1676	I, 128; II, 119, 257, 275; IV, 506
		1686 .... *P*	V, 231, 313; III, 455
$\int$	Integral	1691*P*	*Ae* (1691), 178, 181; (1692), 276; V, 131, 275; VI, 144
$\int\overline{\int}$	Integral of an integral	1675	*E*, 125
$\int \overline{\int \overline{l}\frac{l}{a}}$	Integral of an integral with vinculum	1675	*E*, 59, 125
$\int x \int y$	"Jam $\int x = \frac{x^2}{2}$"	1675	*E*, 137
$\int dv \int d\psi$	....	1675	*E*, 138
$d\int x$ *aequ.* x	$d(\int x) = x$	?	*E*, 153
$\int^n$	nth sum or integral	1695	II, 301; III, 221

545. *Equality.*—

Sign	Meaning	Year	Reference
=	Equal	1666*P* 1675	V, 15 *E*, 132–47
$\sqcap$	Equal	1675	*B*, XII, XIII, 151–60; *E*, 118, 121–31
		1676	I, 101, 154, 160, 163; II, 11; IV, 477; V, 92, 154; VII, 141, 153, 251
aeq. *aequ.*	Equal	1679 .... 1684*P*	I, 31; IV, 485; VII, 47, 88; *E*, 150–55 V, 220
=	Equal	1675	I, 117; II, 60, 76, 116–19, 196, 222, 239; III, 167; IV, 337; V, 112, 241; VII, 55, 266; *E*, 132
=	Equal	1684 .... *P*	V, 125, 222, 256, 279; VI, 156; VII, 222
$\underset{=}{(n)}$	nth equation as in "$nG\overset{(7)}{=}\int\overline{g}$"	? 1694*P*	VI, 427; VII, 359, 364–68, 370, 371, 385, 389 V, 305, 315, 323
$\overset{(n)}{\sqcap}$	nth equation	1675	*E*, 117, 118
$\vert$	Equality as in $g\vert 7$ or $b+c\vert 2+3$	?	VII, 83
ꝏ	Equal	1675?	*Ph.*, VII, 214
∞	Equal	1675?	*Ph.*, VII, 228

546. *Plus, minus.*—

Sign	Meaning	Year	Reference
$+$	$+$	1666 . . . . *P*	V, 15
$-$	$-$	1666 . . . . *P*	V, 15
$\pm$	Plus or minus	1673	*O*, 100, 126; I, 117
$\pm, \pm$	The upper signs simultaneously, or the lower simultaneously	1684*P*	I, 27
$\pm, \mp$		1684*P*	V, 222; VII, 56
$\dotplus, \dotplus$		?	VII, 145
$\mp, \mp$		?	VII, 139, 153
$(\pm), (\pm)$		1684*P*	V, 222
$\pm, (\pm), ((\pm))$	Three independent ambiguities in sign	1684*P*	V, 222
$\pm, \mp, (\mp)$		?	V, 169, 170
$(+)$	Logical combination	?	V, 198
$\oplus$	Logical combination	?	VII, 261, 285
-1 or $\bar{1}$	"Minus one" (sign of quality)	?	V, 370

547. *Multiplication.*—

Sign	Meaning	Year	Reference
. . . .	Apposition	1710 . . . . *P*	VII, 219
$\frown$	Multiplication	1666*P*	V, 15, 20, 24, 25
		?	VII, 33, 45, 49, 54, 100, 220
		1672	I, 29, 156
		1675	*E*, 128, 138, 139
		1677	*B*, 242
$\times$	Multiplication	1676	I, 117; *B*, 242
,	Multiplication	1674	*Cat. crit.*, No. 775; I, 100; VI, 228
		1710*P*	VII, 219
$\cdot$	Multiplication	1676	I, 128; II, 196, 239; III, 36, 160, 341; IV, 211; V, 111, 112; VII, 54, 103, 175
$\cdot$	Multiplication	1684, 1691*P*	V, 129, 222; VII, 219
;	Multiplication and separation as in 2^3; 1.2.3	1713	III, 986
$\divideontimes$	Multiplication	1705	IV, 211

548. *Division.*—

Sign	Meaning	Year	Reference
$\frac{a}{b}$	Fraction, division	1666 . . . . *P*	V, 48, 49, 220
		1673	I, 46, 116; IV, 211; VII, 54, 170
$\smile$	Division	1666*P*	V, 15, 20
		1675	*E*, 134; III, 526; VII, 91, 103
:	Division, ratio	1676	I, 90, 128; IV, 211, 400; VII, 170
		1684 . . . . *P*	V, 131, 223, 228

549. *Proportion.*—

Sign	Meaning	Year	Reference
$: \; :: \; :$	Proportion	1676	I, 146; II, 44, 97, 121; V, 241; VII, 309, 310, 356, 364, 378; *E*, 150
		1684*P*	V, 225
$: = :$	Proportion	1691	II, 113, 118, 119, 196; III, 80, 84, 262, 288, 526; IV, 211; VII, 390, 17 (Wolf)
		1694*P*	V, 315
		1710*P*	VII, 222
$\frac{a}{b}=\frac{c}{d}$	Proportion	1696	III, 262; IV, 211
		1710*P*	VII, 222
$\div\!\!\div$	Continued proportion	1710*P*	VII, 222

Decimal fractions.—

Sign	Meaning	Year	Reference
3.14159, etc.	....	1682*P*	V, 119, 120
		?	VII, 361

550. *Greater or less.*—

Sign	Meaning	Year	Reference
$\sqcap$	Greater	1679	V, 153; VII, 55, 81
$\sqcap$	Less	1679	V, 153, 198; VII, 55, 81
$\overline{=}$	Greater	1710*P*	*Miscell. Berolinensia*
$\underline{=}$	Less	1710*P*	*Ibid.*
$>$	Greater	1749*P*	2d ed. of *ibid.*
$<$	Less	1749*P*	2d ed. of *ibid.* In reprint of article of 1710
$\sqcup$	Greater	1863*P*	VII, 222
$\sqcup$	Less	1863*P*	VII, 222

551. *Aggregation of terms.*—

Sign	Meaning	Year	Reference
$\overline{\qquad}$	Vinculum placed above expression	1675	*E*, 121, 124
		1676	I, 128; VI, 301; VII, 55, 80
		1684 .... *P*	V, 130, 131, 220, 325; VI, 144
$\underline{\qquad}$	Vinculum placed beneath expression	?	VII, 83
$\overline{\underline{n\|}e+fz}^{h}$	Vinculum and *n*th power	1676	I, 116, 128
		1695	III, 175
		1694 .... *P*	V, 313, 323
$\sqrt{\qquad}$	Vinculum and radical sign	1675	I, 76, 120, 156; II, 43; V, 408; VII, 55
		1694 .... *P*	V, 313, 319, 360
$\overbrace{\qquad}$	Aggregating terms beneath the brace	1678*P*	V, 117
$\underbrace{\qquad}$	Aggregating terms above the brace	1678*P*	V, 117
		1677 .... ?	I, 161; VII, 84, 90, 95, 97, 147
$\sqrt{}(\quad)$	Parentheses and radical sign	1679 ... ?	II, 31, 61, 66, 195
$\int(\quad)$	Parentheses and sign of integration	1690	II, 43
$\boxed{n}(\quad)$	Parentheses and *n*th power	1701	III, 662
$\boxed{\ }(\quad)$	Parentheses and fractional power	1701	III, 663
$((\quad))$	Parentheses within parentheses	1696	III, 276
$(\quad)$	Parentheses	1715	IV, 400
$\sqrt{(\quad)}$	Redundant notation	1708?	IV, 356
$A^\frown A-1,,\smile 2$	$A(A-1)\div 2$	1666*P*	V, 20
$1-h+m, :h$	$\frac{1-h+m}{h}$	1676	I, 128
$m, m-n$	$m(m-n)$	1676	I, 100
$,e-,f:g,,$	$(e-(f\div g))$	1696	III, 276
$a+,b:c,,$	$\left(a+\frac{b}{c}\right)$	1696	III, 276
$a+b,:,c+d$	$\frac{a+b}{c+d}$	?	VII, 55
$x.x-1.x-2$	$x(x-1)(x-2)$	1695	III, 160; VII, 101, 103
$\sqrt{}, 2bx-xx$	$\sqrt{2bx-x^2}$	1696	III, 288
$eh+d,:h=e+d(:h)$ ..	$(eh+d)\div h=e+\frac{d}{h}$	1705	IV, 337
$\sqrt{}.,a+b,:,c+d,.,:,l+m$	$\sqrt{\frac{a+b}{c+d}}\div(l+m)$	?	VII, 55
$\sqrt{}((a+b):(c+d)):(l+m)$	$\sqrt{\frac{a+b}{c+d}}\div(l+m)$	?	VII, 55
$\sqrt{,,x:,2b-x}$	Redundant notation	1696	III, 288

552. *Powers and roots.*—

Sign	Meaning	Year	Reference
z^2, x^4	z^2, x^4	1675	*E*, 118
x^3, z^3	x^3, z^3	1676 . . . . *P*	I, 89; V, 129
aa, xx	a^2, x^2	?	*E*, 151
		1684 . . . . *P*	V, 222, 130
x^{a-b}	x^{a-b}	1684*P*	V, 222
y^{-1}	$\frac{1}{y}$	1684*P*	V, 222
$x^{-1:2}$	$x^{-\frac{1}{2}}$	1687?	V, 241
$x^{\frac{n}{\cdot}}, x^{\frac{n-1}{\cdot}}$	x^n, x^{n-1}	?	V, 106
$v^{\frac{1:n-1}{\cdot}}$	$\frac{1}{v^{n-1}}$	1676?	I, 128
$b^{\frac{t}{\cdot}}$	b^t, t variable	1691	II, 76
x^y+y^x	x^y+y^x	1676?	I, 160
$\boxed{2}\overline{r^2+z^2}$	$(r^2+z^2)^2$	1676	I, 116
$\boxed{3}(AB+BC)$	$(AB+BC)^3$	1710*P*	VII, 220
		1701	III, 662, 663
$\overline{\overline{w-e}:f}\boxed{1:h}$	$\left(\frac{w-e}{f}\right)^{\frac{1}{h}}$	1676	I, 128
$\overline{1+y}\rvert^{\frac{2}{3}}$	$(1+y)^{\frac{2}{3}}$	1677	*B*, 242
$\boxed{m}\overline{y+a}$	$(y+a)^m$	1695*P*	V, 323
$\boxed{n}\sqrt[m]{y}$	$y^{\frac{n}{m}}$	?	VII, 184
$\boxed{p^e}y$	y^e	1697	IV, 25
$\overline{a+b}^3$	$(a+b)^3$	?	VII, 55
$\sqrt[2]{}, \sqrt[3]{}$	$\sqrt{}, \sqrt[3]{}$	1676 . . . ?	I, 156; II, 90, 119, 275
$\sqrt[e]{}$ ☽	eth root of polynomial ☽	1705	III, 104
$\left.\frac{20}{1+20}\right\}\boxed{3}, \frac{5}{20}$	$\left(\frac{20}{1+20}\right)^3\cdot\frac{5}{20}$	?	VII, 136
$\surd\boxed{3}$	$\sqrt[3]{}$	1710*P*	*Miscell. Berolinensia*
$\surd$ ③	$\sqrt[3]{}$	1863	VII, 220, 221
$\sqrt{(3)}$	$\sqrt[3]{}$	?	VII, 139, 145

553. *Pure imaginary.*—

Sign	Meaning	Year	Reference
$\sqrt{-\frac{3}{4}}$	$\sqrt{-\frac{3}{4}}$	1675	I, 76
$\surd -3$	$\sqrt{-3}$	1693	II, 236
$\sqrt{-1}$	$\sqrt{-1}$	1675	I, 87
		1695 . . . . *P*	V, 326
$\surd -1$	$\sqrt{-1}$	1674?	II, 11
$\sqrt[2]{-1}, \sqrt[4]{-1}$	$\sqrt{-1}, \sqrt[4]{-1}$	1703	III, 80, 905
$\sqrt[16]{-1}$	$\sqrt[16]{-1}$	1703	III, 80
$\sqrt{(-\sqrt{-1})}$	$\surd(-1\surd(-1))$	1702*P*	V, 360

554. *Logarithm.*—

Sign	Meaning	Year	Reference
log................	Logarithm to any base b	1690? 1691 .... P	II, 53 V, 130, 324
l..................	Logarithm to any base b	1694 1691P	III, 153 V, 129

555. *Trigonometry.*—

Sign	Meaning	Year	Reference
t..................	Tangent	1691	V, 129
s..................	Sine	1691	V, 129; VII, 57
v..................	Versed sine	1691 ...	V, 129; VII, 57
a..................	Arc	1691	V, 129
r..................	Radius	?	VII, 57

556. *Infinite series.*—

Sign	Meaning	Year	Reference
etc. in infinitum......	Series is infinite	1682 .... P	V, 121
etc.	Series is infinite	1675	II, 63, 275; VII, 161, 128
etc.	Series is infinite	1691 .... P	V, 129, 323
etc.	Continued fraction is infinite	1696	III, 351; VII, 47, 48

557. *Similar, Congruent.*—

Sign	Meaning	Year	Reference
∝	Coincident	1679	V, 150, 165
ȣ	Congruent	1679	V, 150, 151, 154, 158, 165; II, 22
$\widehat{E.\widehat{A}.\widehat{C}}$ ȣ $\widehat{F.\widehat{A}.\widehat{C}}$	I.e., $E.A$ ȣ $F.A$, $E.C$ ȣ $F.C$, $A.C$ ȣ $A.C$	1679	V, 159, 160
∽	Similar	1679	V, 153, 154, 172, 185; VII, 57, 222, 277
≌	Congruent	....	V, 172, 173, 185; VII, 222
≌	Alg. identity	1714?	V, 406
|≌|	Coincident	?	V, 185
$\overset{\backsim}{\underset{\backsim}{=}}$	Coincident	?	V, 173
∽	Similar	1710P	*Miscell. Berolinensia*
≌	Congruent	1710P	*Miscell. Berolinensia*
∝	Congruent	?	VII, 263
∞	Coincident	?	VII, 57, 263
∞	Infinite	?	VII, 88
$\overset{\backsim}{r;s;v}$	Relation between radius (r), sine (s), and versed sine (v)	?	VII, 57
$\overset{\backsim}{x;a}$	....	1697	III, 453
a:b;c ∽ l;m;n	$a:b=l:m$, $a:c=l:n$, $b:c=m:n$	1710P	VII, 222
a;b;c::l;m;n.........	"Same relations" as when $a^2+ab=c^2$, $l^2+lm=n^2$	1710P	VII, 222

558. *Function symbol.*—

Sign	Meaning	Year	Reference
ξ	Function of x, ξ being the Greek letter corresponding to x	1698?	III, 537
$\overline{x}\rceil\underline{1}\rfloor$	Function of x	1698?	III, 537
$\overline{x}\rceil\underline{2}\rfloor$	A second function of x	1698?	III, 537
$\overline{x;y}\rceil\underline{1}\rfloor$	Function of x and y	1698?	III, 537
$\overline{x;y}\rceil\underline{2}\rfloor$	Second function of x and y	1698?	III, 537
$\overline{x}\rceil\underline{r.1}\rfloor$	A rational function of x	1698?	III, 537
$\overline{x}\rceil\underline{r.2}\rfloor$	Second rational function of x	1698?	III, 537
$\overline{x}\rceil\underline{ri.1}\rfloor$	Rational integral function of x	1698?	III, 537
$\underset{..}{ab}$	Symmetric function $ab+ac+ad+\ldots.$	?	VII, 178, 191; *Bm.*, XIII, 37, 38
$a_{\cdot}bc$	Symmetric function $abc+abd+\ldots.$	?	
$\underset{..}{a^2}$	Symmetric function $a^2+b^2+\ldots.$	?	
$\underset{..}{a^3}$	Symmetric function $a^3+b^3+\ldots.$	?	

559. *Index numbers, determinant notation.*—

Sign	Meaning	Year	Reference
$x=10+11v+12v^2$, etc.... $y=20+21v+22v^2$, etc..... $z=30+31v+32v^2$, etc.....	These suppositional or fictitious numbers mark rows and columns	1693	II, 239; VII, 161
$\underset{..}{10}.22$	$10.22+12.20$	?	VII, 164
$11.\underset{..}{21}.30$	$11.21.30+11.20.31+10.21.31$ Hence $xy=10.20+1\underset{\cdot}{0}.\underset{\cdot}{2}1v+1\underset{\cdot}{0}.\underset{\cdot}{2}2v^2+\underset{\cdot}{1}3.\underset{\cdot}{2}0v^3+\ldots.$	?	VII, 164
$1_0\,2_1\,3_2$ $1_1\,2_2\,3_0$ $1_2\,2_0\,3_1$	....	1678	VII, 7, 8; *Bm*, XIII, 255
$1_0{}^2{}_1{}^3{}_2$	....	1684	*Bm*, XIII, 255
$10\cdot21\cdot32$ $11\cdot22\cdot30$ $12\cdot20\cdot31$	....	1693	II, 239; *Bm*, XIII, 255
$0=100+110x+101y+111xy+120xx+102yy$ $0=200+210x+201y+211xy+220xx+202yy$		1705	IV, 269
$122.22+133.33+123.23+12.2+13.3+$ 1a *aequ.* 0 $222.22+233.33+223.23+22.2+23.3+$ 2a *aequ.* 0		1675	*Bm*, XIII, 254

560. *Astronomical symbols.*—

Sign	Meaning	Year	Reference
☉	For algebraic expression or equation	1694	III, 142; VII, 47
♀, ☽, ♃, ♂		1705	III, 100–108
☉, ☽		1679	V, 174; VII, 31, 128, 147, 383
☽, ☉, ☿, ♂, ♄, ♃, ♀		?	VII, 390
☉, ☽	To mark places in operations	1703*P*	VII, 224, 225, 236
+☉+☽ *aequ.* ♀ (22)	Here (22) means equation No. 22	1679	V, 170

561. *Marking points, lines, surfaces.*—

Sign	Meaning	Year	Reference
$1B, 2B, 3B$	Points on same line	1677	*Bm*, XIII, 252
$1\ \circledcirc, {}_1 1D$	Double indices marking a surface	1677	*Bm*, XIII, 252
$\overline{X}, \overline{Y}$	Segments of lines	?	V, 185
$Y, (Y)$, etc.	Different positions of a moving point	1679	II, 23, 97
$X, (X)$	Different positions of a moving point	1679	II, 23, 97
$D_2M_3, M_2L, \odot M_2$	Curves	1689	VI, 189
$2G, 3G$	Bodies represented by small circles	1690*P*	VI, 196
$\overline{Y}$	All points of a *locus*	1679	V, 166
$\overline{y}b$	Line traced by moving point		
$\overline{xy}b$	Surface traced by moving curve	1679	V, 148, 149
$\overline{xyz}b$	Solid traced by moving surface		
${}_1C, {}_2C, {}_3C$	Points on a curve	1681	IV, 493; V, 100
${}_1H_1l, {}_2H_1d_3l$	Curves	1684*P*	V, 130
${}_1F, {}_2F, {}_3F$	Positions of moving point	1695	III, 219
$[D], (D), D$	Curve D (D) $[D]$	1688	V, 238
${}_1N_1\xi, {}_3N_3\xi$	Line marked by points	1691 .... *P*	V, 245, 296
${}_1A, {}_2A, {}_2B$	Line marked by points	1696	III, 240; VI, 223, 248, 261, 263, 469; VII, 58
$B(B), F(F), L(L)$	Curves marked by points	1704	III, 741
${}_1L_2L, {}_1B_2B$	Curves marked by points	1706	III, 793
.... anguli ${}_2T\ \odot_3 M$.	Angle, vertex at $\odot$	?	VI, 263
$1_3b, 2_3b, 3_3b$	Point $3b$ on a line takes these positions when line is moved	1679	*Bm*, XIII (1912), 252
$AY\smile A$	AY and then along the arc to A	1714	V, 401

562. *Continued fraction.*—

Sign	Meaning	Year	Reference
$a+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\text{etc.}$	Continued fraction	1696	III, 351, 352; IV, 24; VII, 47
$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+}}}}}}$ etc. *Aequ.* $\odot$	....	?	VII, 47

REMARKS ON TABLES

563. *Plus and minus.*—About 1674, in an article entitled "De la Methode de l'universalité," Leibniz used an involved symbolism which cannot be readily reproduced and explained in our tables. In one of his manuscripts (O 100–104) occurs $\mp a - \mp y$, which means $+a-y$, or $-a+y$; ${}^{+}_{\mp}AB\overset{+}{\underline{\mp}}BC$, which means $+AB-BC$, or $-AB+$

BC, or $+AB+BC$; $\overline{(\mp)\mp AB(\mp)}\underline{\mp}BC$, which means $+AB+BC$, or $-AB+BC$, or $+AB-BC$, or $-AB-BC$; $\underbrace{\mp+a}\underbrace{+\mp}b$, which means the same as $\mp a \div b$, namely, $+a+b$, or $-a+b$, or $+a+b$, or $+a-b$. He gives also some other forms, still more complicated, but he does not use these signs again in later manuscripts.

Multiplication.—Leibniz used both the comma and the dot for multiplication, or for multiplication and aggregation combined. Mahnke pointed out in *Isis* (Vol. IX [1927], p. 289–290) that at first Leibniz used the point and comma, not as signs for multiplication, but simply as signs of separation, and that in this practice he followed Pascal. In his *Dissertatio de arte combinatoria* of 1666 Leibniz (V, p. 40, 44, 51, 60) writes $2.2.2\frown$ for our $2\times2\times2$. These dots are simply signs of separation, for Leibniz writes also $4.18.28.15+$ to mark the sum $4+18+28+15$. Mahnke says that as late as 1673, in unpublished manuscripts, Leibniz writes $3\frown1.+10\frown2.=3\frown3.+2\frown7$, though by that time he used the dot frequently for multiplication. Since the time of Leibniz the dot gained the ascendancy over the comma as the sign of multiplication, while parentheses now generally serve the purpose of aggregation of terms.

564. *Greater and less.*—Reprints of the article "Monitum" in the *Miscellanea Berolinensia* of 1710 contain symbols for "greater" and "less" different from those of 1710. These changes are made without calling the reader's attention to them. Thus in the second edition of the volume of 1710, brought out at Berlin in 1749, the inequality symbols of Harriot are substituted; C. I. Gerhardt's reprint of the article of 1710 brought out in 1863 contains symbols which resemble those of Oughtred, but are not identical with them. We add that Leibniz' signs for "greater" and "less," as given in 1710, closely resemble the signs used by his teacher, E. Weigel.[1] Of the two unequal parallel lines, Weigel's shorter line is drawn the heavier, while Leibniz' longer line is drawn the heavier.

565. *Similar and congruent.*—The first appearance in print of the signs $\backsim$ and $\backsimeq$ for "similar" and "congruent," respectively, was in 1710 in the *Miscellanea Berolinensia* in the anonymous article "Monitum," which is attributed to Leibniz. This form of the symbols was retained in the reprint of the volume of 1710, which appeared in 1749. But in 1863 C. I. Gerhardt reprinted the article in his edition of the mathematical works of Leibniz, and he changed the symbols to $\sim$

[1] *Erhardi Weigeli .... Philosophia mathematica Theologica ....* (Jena, 1693), *Specimina novarum inventionum. I Doctrinalia*, p. 135.

and $\simeq$, respectively, without offering any explanation of the reason for the changes. Both forms for "similar," viz., $\backsim$ and $\sim$, have retained their places in some of our modern textbooks.

Marking points, lines, surfaces.—Eneström, in *Bibliotheca mathematica* (3d ser.), Vol. XI (1910–11), p. 170, 171, shows that notations like $1C$, $2C$, $3C$ were used, before Leibniz, in Fr. van Schooten's Latin edition of Descartes' *Géométrie* (1649), where (p. 112) three points of a curve are marked in a figure C, $2C$, $3C$, while other points in the figure are marked $2S$, $3S$, $2T$, $3T$, $2V$, $3V$. In fact, such notations were used by Descartes himself, in the first (1637) edition of his *Géométrie*, pages 406 and 408. These designations occur also in the *Acta eruditorum* (1682), Plate IV; *ibid.* (1684), page 322; and in Newton's *Principia* (1687), pages 481–82.

DIFFERENTIAL AND INTEGRAL CALCULUS

1. Introduction

566. Without a well-developed notation the differential and integral calculus could not perform its great function in modern mathematics. The history of the growth of the calculus notations is not only interesting, but it may serve as a guide in the invention of fresh notations in the future. The study of the probable causes of the success or failure of past notations may enable us to predict with greater certainty the fate of new symbols which may seem to be required, as the subject gains further development.

Toward the latter part of the eighteenth century the notations of the differential and integral calculus then in vogue no longer satisfied all the needs of this advancing science, either from the standpoint of the new fundamental conceptions that came to be advanced or the more involved analytic developments that followed. Some new symbols were proposed soon after 1750, but the half-century of great unrest and much experimentation in matters of calculus notations was initiated by the publication of J. L. Lagrange's *Théorie des fonctions analytiques*, in 1797.

The fourth volume of Cantor's *Vorlesungen über Geschichte der Mathematik,* dealing with the period 1759–99, pays comparatively little attention to these notations. Nor has anyone yet published a detailed history of calculus notations for the nineteenth century. The present exposition aims to supply a want in this field. Except for certain details, the writer will touch lightly upon the early history of these notations, for that is given quite fully in our histories of mathematics and in the biographies of Newton and Leibniz.

2. Symbols for Fluxions, Differentials, and Derivatives

a) total differentiation during the seventeenth and eighteenth centuries

567. *I. Newton.*—Newton's earliest use of dots, "pricked letters," to indicate velocities or fluxions is found on a leaf dated May 20, 1665; no facsimile reproduction of it has ever been made.[1] The earliest printed account of Newton's fluxional notation appeared from his pen in the Latin edition of Wallis' *Algebra*,[2] where one finds not only the fluxionary symbols $\dot{x}$, $\ddot{x}$, ... , but also the fluxions of a fraction and radical, thus $\overset{\cdot}{\overline{\frac{yy}{b-x}}}$, $\overset{\cdot}{\overline{\sqrt{aa-xx}}}$.

Newton explained his notation again in his *Quadratura curvarum* (London, 1704), where he gave $\overset{||}{x}$, $\overset{|}{x}$, x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, each of these terms being the fluxion of the one preceding, and the fluent of the one that follows. The $\overset{|}{x}$ and $\overset{||}{x}$ are fluent notations. His notation for the fluxions of fractions and radicals did not meet with much favor because of the

[1] S. P. Rigaud, *Historical Essay on the First Publication of Sir Isaac Newton's "Principia"* (Oxford, 1838), Appendix, p. 23. Consult the remarks on this passage made by G. Eneström in *Bibliotheca mathematica* (3d ser.), Band XI (Leipzig, 1910–11), p. 276, and Band XII (1911–12), p. 268, and by A. Witting in Band XII, p. 56–60. See also *A Catalogue of the Portsmouth Collection of Books and Papers, written by or belonging to Isaac Newton* (Cambridge, 1888), p. xviii, 2. The use of dots before the time of Newton, in England, for the specialized purposes of indicating comparatively small quantities, is evident by reference to Leonard Digges and Nicolas Mercator. Digges said, in his Booke named *Tectonicon* (London, 1592), p. 15, where he explains his Table of Timber Measure, "Supposing the square of your Timber were 7. Inches, and that ye desired to know what measure of length of the Ruler would make a foot square: Seeke in the left margine seven Inches: and with him in that order toward the right hand, ye shal find 2. foote, 11. Inches, and $\frac{2}{7}\cdot$ of an Inch. Note because the fraction $\frac{\cdot 2}{7}$ hath a prick by him, it betokeneth some small quantitie lesse then $\frac{2}{7}\cdot$ of an Inch. If it had 2. prickes or points thus $:\frac{2}{7}$ it should signifie som litle quantitie more. Neither maketh it matter whether ye observe this pricking or no, the quantitie is so little to be added or pulled away." It is of some interest that in 1668, before Newton's notation had appeared in print, N. Mercator used the dot over the letter I, thus $\dot{I}$, to signify an infinitesimal, and over numbers, $\dot{6}4$ for example, to serve as a reminder that 64 was the coefficient of a power of an infinitesimal. See *Philosophical Transactions* (London), Vol. III, p. 759 ff. Reprinted in Maseres, *Scriptores logarithmici*, Vol. I, p. 231. See also *Nature*, Vol. CIX (1922), p. 477.

[2] J. Wallis, *De algebra tractatus* (Oxoniae, 1693), p. 390–96.

typographical difficulties. In another place[1] he writes "$\dot{z}:\dot{x}$", which in modern notation would be $\frac{dz}{dt}:\frac{dx}{dt}$, or the derivative $\frac{dz}{dx}$.

English writers of the eighteenth century used the Newtonian dots almost exclusively, but sometimes with slight variations. Thus H. Ditton[2] and J. Clarke extended the use of two dots separated by a vinculum to denote the first fluxion. They explain: "$l:\overline{x+a}^{\,m}$ denotes the Log. of $\overline{x+a}^{\,m}$, and $l:\overset{\div}{\overline{x+a}}{}^{m}$ the Fluxion of that Log. Also $l:x+\overset{m}{a}$ denotes the Log. of the Quantity x added to the m Power of a, and $l:\overset{\div}{x}+\overset{m}{a}$ the Fluxion of the Log. of the same."

568. The Newtonian symbols are found also on the European Continent. Thus the Frenchman, Alexis Fontaine,[3] uses $\dot{x}$, $\dot{y}$, $\dot{s}$ without defining the notation except by $\overset{|}{x}-x=\dot{x}$, $\overset{|}{y}-y=\dot{y}$, where x, y and $\overset{|}{x}$, $\overset{|}{y}$ are points on a curve, not far apart, and $\dot{s}$ is the arc from (x, y) to $(\overset{|}{x}, \overset{|}{y})$. The symbols are not used as velocities, but appear to be small constant or variable increments or variations, which may be used by the side of the Leibnizian differential "d," thus $d\dot{s}=\frac{y\,d\overset{|}{y}}{\dot{s}}$. In deriving the curve of quickest descent, he ends with the statement: "*Fluxion* $\frac{\dot{y}}{x^{\frac{1}{2}}\dot{s}}=0$, dont la *Fluente* est $\frac{\dot{y}}{x^{\frac{1}{2}}\dot{s}}=\frac{1}{a^{\frac{1}{2}}}$, que l'on trouvera être l'équation de la cycloide."

In a paper dated 1748 Fontaine[4] treats differential equations and uses $\dot{x}$, $\dot{y}$ in place of dx, dy. Differentiating $n(1-n)p+2nx+(2-5n)y=0$, he obtains $2n\dot{x}+(2-5n)\dot{y}=0$; eliminating n, he arrives at "l'équation aux premières différences $3p\dot{y}^2+10x\dot{y}^2-10y\dot{x}\dot{y}-2p\dot{x}\dot{y}-4x\dot{x}\dot{y}+4y\dot{x}^2=0$." The Newtonian dot was used also by a few other continental writers of the eighteenth century and were found regularly in the republications on the Continent of the works of Newton and of a

[1] Isaaci Newtoni, *Opera* (ed. S. Horsley; London), Vol. I (1779), p. 410.

[2] Humphrey Ditton, *An Institution of Fluxions* (2d ed. by John Clarke; London, 1726), p. 117.

[3] *Mémoires donnés à l'académie royale des sciences, non imprimés dans leur temps*. Par M. Fontaine, de cette Académie (Paris, 1764), p. 3. Fontaine uses the dot also in articles in *Histoire de l'académie r. des sciences*, année 1734 (Paris, 1736), *Mémoires*, p. 372; année 1767 (Paris, 1770), p. 588, 596. For further data see F. Cajori, "Spread of Calculus Notations," *Bull. Amer. Math. Soc.*, Vol. XXVII (1921), p. 455.

[4] Fontaine, *op. cit.*, p. 85.

few other Englishmen, among whom are John Muller, David Gregory, and Colin Maclaurin.[1]

After the middle of the eighteenth century, when feelings of resentment ran high between the adherents of Leibniz and their opponents, it is extraordinary that a mathematical journal should have been published on the European Continent for fifteen years, which often uses the Newtonian notation, but never uses the Leibnizian. The journal in question is a monthly, published at Amsterdam from 1754 to 1769. It appeared under the title, *Maandelykse Mathematische Liefhebbery* ("Monthly Mathematical Recreations"). It contains problems on maxima and minima, solved with the aid of fluxions dressed in the familiar garb of the Newtonian dots. The first twelve volumes of the journal give altogether forty-eight or more such problems; the last few volumes contain no fluxions. Among the contributors using fluxions were G. Centen, J. Schoen, J. Kok, J. T. Kooyman, F. Kooyman, A. Vryer, J. Bouman, D. Bocx. The two most noted were P. Halcke and Jakob Oostwoud. The latter was a teacher of mathematics in Oost-Zaandam near Amsterdam, who in 1766 became a member of the Hamburg Mathematical Society.[2] Later he collected and published the problems solved in his journal in three separate volumes.[3]

Among other Dutch adherents of the Newtonian notation may be cited P. Steenstra[4] and A. B. Strabbe.[5] Strabbe's text on the calculus uses the notation of Newton exclusively and develops the subject in the manner of the English mathematicians of the eighteenth century.

569. *Quotation from Newton.*—This quotation is of interest because it is Newton's own statement regarding his invention and use

[1] It should be stated, however, that in the Introduction and annotations to Newton's *Principia*, brought out by Thomas le Seur and François Jacquier at Cologne, in 1760, the Leibnizian notation is employed by the editors.

[2] Bierens de Haan in *Festschrift, herausgegeben von der Mathematischen Gesellschaft in Hamburg, anlässlich ihres 200 jährigen Jubelfestes 1890*, Erster Teil (Leipzig, 1890), p. 79.

[3] Bierens de Haan, *op. cit.*, p. 80, where the titles of the three publications are given in full. See also De Haan's *Bouwstoffen voor de Geschiedenis der Wis-en Natuurkundige Wetenschappen in de Nederlanden* (1878), p. 76–85, reprinted from *Verslagen en Mededeelingen der Kon. Akademie van Wetenschappen*, Afd. Natuurk., 2e Reeks, Deel VIII, IX, X, en XII.

[4] P. Steenstra, *Verhandeling over de Klootsche Driehoeks-Meeting* (Amsterdam, 1770), p. 167.

[5] Arnoldus Bastiaan Strabbe, *Erste Beginselen der Fluxie-rekening* (Amsterdam, 1799).

of fluxional notations. It occurs in a review of John Collins' *Commercium epistolicum* bearing on the controversy between Newton and Leibniz on the invention of the calculus. In that review, which appeared in the *Philosophical Transactions*,[1] Newton refers to himself in the third person: "Mr. Newton doth not place his Method in Forms of Symbols, nor confine himself to any particular Sort of Symbols for Fluents and Fluxions. Where he puts the Areas of Curves for Fluents, he frequently puts the Ordinates for Fluxions, and denotes the Fluxions by the Symbols of the Ordinates, as in his *Analysis*. Where he puts Lines for Fluents, he puts any Symbols for the Velocities of the Points which describe the Lines, that is, for the first Fluxions; and any other Symbols for the Increase of those Velocities, that is, for the second Fluxions, as is frequently done in his *Principia philosophiae*. And where he puts the Letters x, y, z for Fluents, he denotes their Fluxions, either by other Letters as p, q, r; or by the same Letters in other Forms as X, Y, Z or $\dot{x}$, $\dot{y}$, $\dot{z}$; or by any Lines as DE, FG, HI, considered as their Exponents. And this is evident by his Book of *Quadratures*, where he represents Fluxions by prickt Letters in the first Proposition, by Ordinates of Curves in the last Proposition, and by other Symbols, in explaining the Method and illustrating it with Examples, in the Introduction. Mr. Leibnitz hath no Symbols of Fluxions in his Method, and therefore Mr. Newton's Symbols of Fluxions are the oldest in the kind. Mr. Leibnitz began to use the Symbols of Moments or Differences dx, dy, dz, in the Year 1677. Mr. Newton represented Moments by the Rectangles under the Fluxions and the Moment o, when he wrote his *Analysis*, which was at least Forty Six Years ago. Mr. Leibnitz has used the symbols $\int x$, $\int y$, $\int z$ for the Sums of Ordinates ever since the Year 1686; Mr. Newton represented the same thing in his *Analysis*, by inscribing the Ordinate in a Square or Rectangle. All Mr. Newton's Symbols are the oldest in their several Kinds by many Years.

"And whereas it has been represented that the use of the Letter o is vulgar, and destroys the Advantages of the Differential Method: on the contrary, the Method of Fluxions, as used by Mr. Newton, has all the Advantages of the Differential, and some others. It is more elegant, because in his Calculus there is but one infinitely little Quantity represented by a Symbol, the Symbol o. We have no Ideas of infinitely little Quantities, and therefore Mr. Newton intro-

[1] *Philosophical Transactions*, Vol. XXIX, for the years 1714, 1715, 1716 (London, 1717), p. 294–307.

duced Fluxions into his Method, that it might proceed by finite Quantities as much as possible. It is more Natural and Geometrical, because founded upon the *primae quantitatum nascentium rationes*, which have a being in Geometry, whilst *Indivisibles*, upon which the Differential Method is founded have no Being either in Geometry or in Nature. There are *rationes primae quantitatum nascentium*, but not *quantitates primae nascentes*. Nature generates Quantities by continual Flux or Increase; and the ancient Geometers admitted such a Generation of Areas and Solids, when they drew one Line into another by local Motion to generate an Area, ...

"In the Year 1692, Mr. Newton at the Request of Dr. Wallis, sent to him a Copy of the first Proposition of the Book of Quadratures, with Examples thereof in the first, second and third Fluxions: as you may see in the second Volume of the Doctor's Works, p. 391, 392, 393 and 396. And therefore he had not then forgotten the Method of Second Fluxions."

570. *G. W. Leibniz.*—In the years 1672–76 Leibniz resided most of the time in Paris and, being influenced strongly by Cartesian geometry, he entered upon the study of quadratures. He divided figures by ordinates into elements and on October 26, 1675, letting $\sqcap$ express equality, wrote down the theorem:[1] "Differentiarum momenta ex perpendiculari ad axem aequantur complemento summae terminorum, sive Momenta Terminorum aequantur complemento summae summarum, sive omn. $\overline{xw} \sqcap$ ult. x, $\overline{\text{omn. } w}$,, $-$omn. $\overline{\text{omn. } w}$. Sit $xw \sqcap a\mathfrak{z}$, fiet $w \sqcap \frac{a\mathfrak{z}}{x}$, fiet omn. $\overline{a\mathfrak{z}} \sqcap$ ult. x omn. $\frac{a\mathfrak{z}}{x}$ $-$omn. $\overline{\text{omn. } \frac{a\mathfrak{z}}{x}}$; ergo omn. $\frac{a\mathfrak{z}}{x} \sqcap$ ult. x, omn. $\frac{a\mathfrak{z}}{x^2}$ $-$omn. omn. $\frac{a\mathfrak{z}}{x^2}$, quo valore in aequ. praecedenti inserto fiet: omn. $a\mathfrak{z} \sqcap$ ult. x^2 omn. $\frac{a\mathfrak{z}}{x^2}$ $-$ult. x, $\overline{\text{omn. } \overline{\text{omn. } \frac{a\mathfrak{z}}{x^2}}}$ $-$ $\overline{\text{omn. } \overline{\text{ult. } x. \text{ omn. } \frac{a\mathfrak{z}}{x^2}} - \overline{\text{omn. } \overline{\text{omn. } \frac{a\mathfrak{z}}{x^2}}}}$. Et ita iri potest infinitum." ("The moments of the differences about a perpendicular to the axis are equal to the complement of the sum of the terms, or, the moments of the terms are equal to the complement of the sum of the sums, that

[1] C. I. Gerhardt, *Der Briefwechsel von G. W. Leibniz mit Mathematikern*, Vol. I (Berlin, 1899), p. xii, xiii; C. I. Gerhardt, Die *Entdeckung der höheren Analysis* (Halle, 1855), p. 120. See also the English translation, J. M. Child, *The Early Mathematical Manuscripts of Leibniz* (Chicago and London, 1920), p. 70–80. See our §§ 538–65.

is, omn. $\overline{xw}\ \sqcap$, ... , and this can proceed in this manner indefinitely.") Referring to Leibniz' figure (Fig. 121), the top horizontal line is the axis, the w's are small parts of the horizontal lines, omn. w is a full horizontal line, x is a variable height having the altitude of the rectangle as its "ultimate" value.

The last equation given above forcibly exhibits the necessity of a simplified notation.

Three days later, October 29, 1675, Leibniz constructs what he called the *triangulum characteristicum*, which had been used before him by Pascal and Barrow, and considers the question whether expressions like "omn. l" and "omn. p," where l and p designate lines, can be multiplied one into the other. Such products can be formed

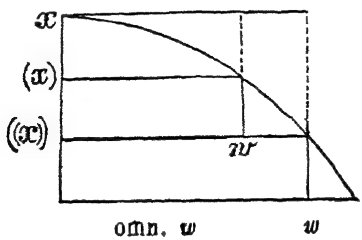


FIG. 121

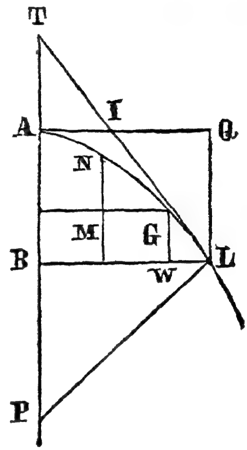


FIG. 122

FIG. 121.—Leibniz' figure in MS dated Oct. 26, 1675. Note also the use of parentheses in designating different x's. (Taken from C. I. Gerhardt, *Die Entdeckung der höheren Analysis* [1855], p. 121.)

FIG. 122.—From the manuscript of Leibniz, Oct. 29, 1675, as reproduced by C. I. Gerhardt, *Entdeckung*, etc., p. 123.

if "omn. l" is a line and "omn. p" is a plane figure; the product is the volume of a solid. In a figure like Figure 122 he lets $BL \sqcap y$, $WL \sqcap l$, $BP \sqcap p$, $TB \sqcap t$, $AB \sqcap x$, $GW \sqcap a$, $y \sqcap$ omn. l, and obtains $\frac{l}{a} \sqcap \frac{p}{\text{omn. } l \sqcap y}$, consequently $p \sqcap \frac{\overline{\text{omn. } l}}{a}\, l$, i.e., omn. l is to be multiplied into l; moreover, omn. $p \sqcap \frac{y^2}{2} \sqcap \frac{\overline{\text{omn. } l\ \boxed{2}}}{2}$; "ergo habemus theorema, quod mihi videtur admirabile et novo huic calculo magni adjumenti loco futurum, nempe quod sit $\frac{\overline{\text{omn. } l\ \boxed{2}}}{2} \sqcap \text{omn. } \overline{\overline{\text{omn. } l}\ \frac{l}{a}}$ qualiscunque sit l." ("Hence we have a theorem that to me seems

admirable, and one that will be of great service to this new calculus, namely. . . . .")

A little further on, in the manuscript dated October 29, 1675, he says: "Utile erit scribi $\int$ pro omn. ut $\int l$ pro omn. l, id est summa ipsorum l. Itaque fiet $\frac{\int \overline{l^2}}{2} \sqcap \int \overline{\int \bar{l} \frac{l}{a}}$ et $\int \bar{x} l \sqcap x \int \bar{l} - \int \int \bar{l}$." ("It will be useful to write $\int$ for omn., as $\int l$ for omn. l, that is, the sum of these l's. Thus, one obtains. . . . .") Here, then, is the origin of the fundamental symbol in the integral calculus.[1] Still further on Leibniz remarks, "Satis haec nova et nobilia, cum novum genus calculi inducant." ("These are sufficiently new and notable, since they will lead to a new calculus.")

It is in the same manuscript dated October 29, 1675, that he remarks,[2] "Si sit $\int l \sqcap ya$. Ponemus $l \sqcap \frac{ya}{d}$ nempe ut $\int$ augebit, ita d minuet dimensiones. $\int$ autem significat summam, d differentiam." ("Suppose that $\int l = ya$. Let $l = \frac{ya}{d}$; then just as $\int$ will increase, so d will diminish the dimensions. But $\int$ means sum, and d a difference.") Ten days later, November 11, 1675,[3] he remarks, "Idem est dx et $\frac{x}{d}$, id est differentia inter duas x proximas." ("dx and $\frac{x}{d}$ are the same, that is, the difference between two proximate x's.")[4] Further on in

[1] Why did Leibniz choose the long letter *s* (*summa*), rather than the letter *o* of the word *omnia* which he had been using? Was it because the long *s* stood out in sharper contrast to the other letters and could be more easily distinguished from them? Sixteen years earlier (in 1659) Pietro Mengoli of Bologna used *o* in summation, in his *Geometriae speciosae elementa*, p. 53, where he says: "massam ex omnibus abscissis, quàm significamus charactere O.a; et massam ex omnibus residuis O.r; et massam ex omnibus abscissis secundis O.a2."

[2] C. I. Gerhardt, *Entdeckung* (1855), p. 126.

[3] C. I. Gerhardt, *Briefwechsel*, p. xiv; C. I. Gerhardt, *Entdeckung*, p. 134. See also our § 532.

[4] Leibniz' representation of the differential of y successively by ω, l, $\frac{y}{d}$, and finally by dy goes against the conjecture that he might have received a suggestion for the notation dy, dx, from Antoine Arnauld who, in his *Nouveaux Elémens de Géométrie* (Paris, 1667), adopts $d'x$, $d'y$ as the designation of a small, aliquot part of x and y. "Could this French $d'x$ have been hovering before Leibniz when he made his choice of notation?" is the question which naturally arose in the mind of Karl Bopp in *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Vol. XIV (Leipzig, 1902), p. 252.

the same manuscript he enters upon the inquiry: "Videndum an $dx\,dy$ idem sit quod $d\,\overline{xy}$, et an $\frac{dx}{dy}$ idem quod d$\frac{x}{y}$." ("Let us see whether $dx\,dy$ is the same as $d\,\overline{xy}$, and whether $\frac{dx}{dy}$ is the same as $d\frac{x}{y}$.") And he concludes that neither pair of expressions is the same. Thus on November 11, 1675, he introduced the symbol dx and dy as the differentials of x and y, and also the derivative form $\frac{dx}{dy}$.

571. The first appearance of dx in print was in an article which Leibniz contributed to the *Acta eruditorum*, in 1684.[1] Therein occur the expressions "dw ad dx," "dx ad dy," and also "dx:dy," but not the form "$\frac{dx}{dy}$." For another symbol for the first derivative, at one time considered by Leibniz, see § 539. Leibniz used the second derivative[2] written in the form $ddx:\overline{dy}^2$. In his *Historia et origo calculi differentialis*, written by Leibniz not long before his death, he lets[3] $dx=1$, and writes $d(x)^3=3xx+3x+1$. In his paper, printed in 1684, dx appears to be finite when he says: "We now call any line selected at random dx" ("Iam recta aliqua pro arbitrio assumta vocetur dx").[4]

Leibniz writes $(dx)^2$ in the form $dx\,dx$; Child remarks,[5] "Leibniz does not give us an opportunity of seeing how he would have written the equivalent of $dx\,dx\,dx$; whether as dx^3 or $\overline{dx}^3$ or (dx).[3]" In the case of a differential of a radical, Leibniz uses a comma after the d, thus:[6] $d, \sqrt[b]{x^a}$, *aeq.* $\frac{a}{b}\,dx\sqrt[b]{x^{a-b}}$. In imitation of Leibniz' $dd\,v$ for the second differential, Johann Bernoulli[7] denotes the fourth differential by $dddd\,z$. Not until 1695, or twenty years after Leibniz' first use of d,

[1] See also *Leibnizens Mathematische Schriften*, Vol. V (Halle, 1858), p. 220–26.

[2] *Acta eruditorum* (1693), p. 179; *Leibnizens Mathematische Schriften*, Vol. V (1858), p. 287.

[3] *Op. cit.*, Vol. V (Halle, 1858), p. 406.

[4] For the various explanations of the infinitesimal given by Leibniz at different times of his career see G. Vivanti, "Il Concetto d'Infinitesimo," *Giornale di Matematiche di Battaglini*, Vol. XXXVIII and XXXIX.

[5] J. M. Child, *Early Mathematical Manuscripts of Leibniz* (Chicago and London, 1920), p. 55.

[6] *Acta eruditorum* (1684), p. 469. *Leibnizens Math. Schriften*, Vol. V (Halle, 1858), p. 222.

[7] *Acta eruditorum* (1694), p. 439; Joh. Bernoulli, *Opera*, Vol. I, p. 125–28.

did he suggest the use of numbers in the writing of the higher differentials; in a letter[1] of October 20/30, 1695, addressed to Johann Bernoulli, he writes the equation showing the relationship between the two symbols, $d^m = \int^n$ when $n = -m$.

In the same year he published a reply to B. Nieuwentijt's[2] attacks upon the calculus, in which he writes ddx seu $d^2 \cdot x$, $dddx$ sive $d^3 \cdot x$.

De l'Hospital,[3] who was the earliest to write a textbook on the calculus, followed Leibniz' original practice and did not use numerals in the designation of the higher differentials; he wrote $dd\,y$, $ddd\,y$, $dddd\,y$. As late as 1764 we find these very notations in the writings of Fontaine.[4] William Hales,[5] an adherent of Newton's fluxions, in 1804, cites the symbols dx, $dd\,x$, $ddd\,x$, $dddd\,x$ and then declares them to be *minùs elegantèr* than the fluxionary symbols.

572. Minute adjustments of notation arose in the eighteenth century. Thus, Euler wrote $d \cdot x\,dy\,dz$ for $d(x\,dy\,dz)$. The same practice is observed by Lacroix; but $dx\,dy\,dz$ always meant the product of the three differentials. Bézout[6] points out that dx^2 means $(dx)^2$, and $d(x^2)$ means the "différentielle de x^2." Lacroix explains[7] that d^2y^2 is the same thing as $(d^2y)^2$ and, in general, d^ny^m indicates $(d^ny)^m$, that $d \cdot x^2$ means the differential of x^2 and $d \cdot uv$ of uv; the point following the d signifying that the operation applies to all that immediately follows.

No substantial changes in the notation for total differentiation were made until after the middle of the eighteenth century. A provisional, temporary notation Δ for differential coefficient or *différences des fonctions* was used in 1706 by Johann Bernoulli.[8] Previously he had used[9] the corresponding Latin letter D. The changes that came

[1] *Leibnizens Math. Schriften*, Vol. III (1855), p. 221.

[2] *Acta eruditorum* (1695), p. 310; *Leibnizens Math. Schriften*, Vol. V, p. 321.

[3] De l'Hospital, *Analyse des infiniment petits* (2d ed.; Paris, 1715), p. 55. The date of the first edition was 1696.

[4] A. Fontaine, *Mémoires donnés à l'académie r. des sciences, non imprimés dans leur temps* (1764), p. 73.

[5] Fr. Maseres, *Scriptores logarithmici*, Vol. V (London, 1804), p. 131.

[6] Etienne Bézout, *Cours de Mathématiques à l'usage du corps de l'artillerie*, Tome III (Paris, 1797), p. 18.

[7] S. F. Lacroix, *Traité du Calcul différentiel et du Calcul intégral*, Tome I (2d. ed.; Paris, 1810), p. 208; also his *Traité élémentaire de Calcul différentiel et de Calcul intégral* (Paris, 1802), p. 9, 18.

[8] G. Eneström, *Bibliotheca mathematica* (2d ser.), Vol. X (1896), p. 21.

[9] *Leibnizens Math. Schriften* (ed. C. I. Gerhardt), Vol. III, p. 531.

were due in most cases to new conceptions of the fundamental operations, conceptions which seemed to demand new symbols.

573. *J. Landen.*—One of the first in the field was John Landen, who published at London in 1758 his *Discourse concerning Residual Analysis* and in 1764 his *Residual Analysis*. These publications mark an attempt at arithmetization of the calculus, but the process was so complicated as to be prohibitive. In his *Mathematical Lucrubations* of 1755, Landen still follows Newtonian procedure and notation, but his *Discourse* of 1758 marks a departure. Still proceeding geometrically, he introduces (p. 12) two new symbols, "$[x|y]$ being put for the quotient of $y-u$ divided by $x-v$," i.e., for $\Delta y:\Delta x$. Then he, in fact, proceeds to the limit, "by taking v equal to x, and writing $[x\perp y]$ for the value of $[x|y]$ in the particular case when v is so taken." This notation $[x\perp y]$ for the first derivative labors under the disadvantage of not being compact and requiring many distinct strokes of the pen, but typographically it enjoys an advantage over $\frac{dy}{dx}$ in not requiring a fractional line and in calling for type of normal height. In considering maxima and minima, on page 24, Landen invents a new designation, "writing $[x \perp\!\perp y]$ for the value of the quotient of $[x\perp y]-[v\perp u]$ divided by $x-v$, when v is therein taken equal to x." Landen had no followers in the use of his symbols.

574. *A. Fontaine.*—In France, Alexis Fontaine introduced a notation for partial differentiation which we shall notice later; he takes $\frac{du}{dx}$ to represent a partial derivative and $\frac{1}{dx}\,du$ a complete derivative.[1] This form of the complete derivative is seldom found later. We have seen a trace of it in Da Cunha,[2] who in 1790 writes the differential

$$d\left(\frac{1}{dx}\,d\,\frac{dy}{dx}\right),$$

and in De Montferrier,[3] who in 1835 writes the derivative

$$\frac{1}{d\Phi\dot{x}}\cdot d\left[\frac{dF\dot{x}}{d\Phi\dot{x}}\right].$$

[1] A. Fontaine, *Mémoires donnés à l'académie r. des sciences, non imprimés dans leur temps* (Paris, 1764), in "Table" of the *Mémoires*.

[2] J. A. da Cunha, *Principios mathematicos* (Lisbon, 1790), p. 244.

[3] A. S. Montferrier, *Dictionnaire des sciences mathématiques* (Paris, 1835), art. "Différence," p. 460.

Here the point placed above the x is to indicate that, after the differentiations, the x is assigned the value which makes $\Phi\, x=0$.

575. *J. L. Lagrange.*—A strikingly new treatment of the fundamental conceptions of the calculus is exhibited in J. L. Lagrange's *Théorie des fonctions analytiques* (1797). Not satisfied with Leibniz' infinitely small quantities, nor with Euler's presentation of dx as 0, nor with Newton's prime and ultimate ratios which are ratios of quantities at the instant when they cease to be quantities, Lagrange proceeded to search for a new foundation for the calculus in the processes of ordinary algebra. Before this time the derivative was seldom used on the European Continent; the differential held almost complete sway. It was Lagrange who, avoiding infinitesimals, brought the derivative into a supreme position. Likewise, he stressed the notion of a function. On page 2 he writes, "Ainsi fx désignera une fonction de x." On page 14 he proceeds to a new notation for the first, second, third, etc., derivatives of fx, with respect to x. He says: "... pour plus de simplicité et d'uniformité, on dénote par $f'x$ la première fonction dérivée de fx, par $f''x$ la première fonction dérivée de $f'x$, par $f'''x$ la première fonction dérivée de $f''x$, et ainsi de suite." "Nous appellerons la fonction fx, *fonction primitive*, par rapport aux fonctions $f'x$, $f''x$, etc., qui en dérivent, et nous appellerons celles-ci, fonctions dérivées, par rapport à celle-là. Nous nommerons de plus la première fonction dérivée $f'x$, *fonction prime;* la seconde dérivée $f''x$, *fonction seconde;* la troisième fonction dérivée $f'''x$, *fonction tierce*, et ainsi de suite. De la même manière, si y est supposé une fonction de x, nous dénoterons ses fonctions dérivées par y', y'', y''', etc."[1]

The same notation for the total derivatives of fx was used by Lagrange in his *Leçons sur le calcul des fonctions* which appeared at Paris in 1801 and again in 1806.[2]

[1] For typographical reasons the Council of the London Mathematical Society in 1915 expressed preference for x', r', θ'' over $\dot{x}$, $\dot{r}$, $\ddot{\theta}$. G. H. Bryan objects to this, saying: "It is quite useless to recommend the substitution of dashes for dots in the fluxional notation for velocities and accelerations, because dashes are so often used for other purposes. For example θ'' might be an angle of θ seconds. The only rational plan of avoiding the printer's difficulties is to place the superior dots *after* the letters instead of over them; thus $x^{\cdot}y^{\cdot\cdot}-y^{\cdot}x^{\cdot\cdot}$ is all that is necessary" (*Mathematical Gazette*, Vol. VIII [1917], p. 172, 221). This last notation is actually found in a few publications at the opening of the nineteenth century (§ 630).

[2] Reprints are not always faithful in reproducing notations. Thus, in the 1806 edition of the *Leçons*, p. 10, 12, one finds $fx, f'x, f''x, f'''x$, while in the *Œuvres*, Vol. X, which purports to be a reprint of the 1806 edition, on p. 15, 17, one finds the corresponding parts given as $f(x), f'(x), f''(x), f'''(x)$.

It should be noticed that the 1797 edition of Lagrange's *Théorie des fonctions* is not the first occurrence of the accent as the mark for derivatives. In 1770 Ψ' occurs for $\frac{d\Psi}{dx}$ in his *Nouvelle méthode pour résoudre les équations littérales,* and in 1759 the notation is found in a part of a memoir by François Daviet de Foncenex in the *Miscellanea Taurinensia,* believed to have been written for Foncenex by Lagrange himself.[1]

576. *J. Pasquich.*—Another writer who at this time attempted a reform of the calculus was Johann Pasquich, of Ofen (Budapest) in Hungary, in a paper entitled "Exponential Rechnung."[2] Postulating that every function can be expressed in the form $y=Ax^a+Bx^b+\ldots.$, he calls the function $\epsilon y=aAx^a+bBx^b+\ldots.$ the *exponential* of y; "εy" is really the limit of $\frac{x\Delta y}{\Delta x}$.

J. P. Grüson.—A change similar to that of Pasquich was suggested by Johann Philipp Grüson, of Berlin, in his *"Calcul d'exposition."*[3] He reaches the same limit which had been used by Pasquich. Grüson indicated it by an inverted and rounded E; thus $\mathcal{3}F$ represented the limit of $\frac{x\Delta F}{\Delta x}$. This notation is mentioned by S. F. Lacroix in his *Traité du Calcul différentiel et du Calcul intégral.*

The notations of Pasquich and Grüson found no acceptance among the mathematicians of their time and have not been drawn upon since for the designation originally assigned. The notation of Lagrange, on the other hand, was adopted by many and has been found a useful adjunct of other calculus notations.

577. *L. F. A. Arbogast.*—Still another symbolism, which has been extensively used in recent years, was made public by L. F. A. Arbogast in his *De Calcul des Dérivations* (Strasbourg, 1800). This new calculus was offered as comprising the theory of series, and included the differential calculus as a special case. An outline of it was presented in the form of a memoir in 1789 to the Academy of Sciences at Paris, but was not published at the time. It was mentioned in the

[1] P. E. B. Jourdain in *Proceed. 5th Internat. Congress* (Cambridge, 1913), Vol. II, p. 540.

[2] J. Pasquich, "Anfangsgründe einer neuen Exponentialrechnung," *Archiv der reinen und angew. Math.*, Vol. II (1798), p. 385–424. Our information is drawn from M. Cantor, *op., cit.* Vol. IV, p. 667, 668.

[3] J. P. Grüson, "Le Calcul d'exposition," *Mém. académie* (Berlin, 1798), Pub. 1801, p. 151–216; (1799 and 1800) Pub. 1803, p. 157–88.

1797 edition of Lagrange's *Théorie des fonctions analytique* as well as in the first volume of S. F. Lacroix's *Traité du Calcul différentiel et du Calcul intégral.* Arbogast expresses himself in his Preface as follows:

"To form the algorithm of derivations, it became necessary to introduce new signs; I have given this subject particular attention, being persuaded that the secret of the power of analysis consists in the happy choice and use of signs, simple and characteristic of the things which they are to represent. In this regard I have set myself the following rules: (1) To make the notations as much as possible analogous to the received notations; (2) Not to introduce notations which are not needed and which I can replace without confusion by those already in use; (3) To select very simple ones, yet such that will exhibit all the varieties which the different operations require."

His principal symbol is the D, as the sign for "derivation." This symbol had been previously used by Johann Bernoulli (§§ 528, 560). During the latter part of the eighteenth century the D had been used by several authors to represent a finite difference. Arbogast lets $F(a+x)$ be any function (*une fonction quelconque*) of the binomial $a+x$; one knows, he says, that one can develop that function in a series proceeding according to the powers of x, viz., $a+bx+\frac{c}{1\cdot 2}x^2+\ldots$, where $a=Fa$. He designates by D the operation upon Fa that yields b, so that $b=DFa$, $c=DDFa$, etc. While Arbogast's symbol D for our derivative has maintained its place in many books to the present time, a large variety of satellites to it, which Arbogast introduced, are now only of antiquarian interest. Placing a dot (p. 2) after the D gives him $D\cdot Fa$ to represent $DFa\cdot D\cdot a$ for cases where $D\cdot a$ is not **1.** He writes (p. 33): "$\underset{c}{D^m}$ au lieu de $\frac{D^m}{1\cdot 2\cdot 3\ldots m}$." On page 308 he states: "Nous désignerons même à l'avenir les coëficiens différentiels $\frac{d\phi x}{dx}$, $\frac{d^2\phi x}{1\cdot 2\cdot dx^2}$, ..., dx étant invariable, par $\partial\phi x$, $\underset{c}{\partial^2}\phi x$, ..., ce qui est la même chose que les dérivées $D\phi x$, $\underset{c}{D^2}\phi x$, ..." Some of his signs are exhibited in Figure 123.

On page 330 Arbogast explains d^{-1}, d^{-2}, ..., and D^{-1}, D^{-2}, ..., as meaning, respectively, *différentielles inverses* and *dérivées inverses*. As an incidental rather than a systematic notation, the d^{-1} and D^{-1} have maintained themselves to the present day. This nota-

tion was used, for instance, by Georg Tralles[1] in 1804, by A. R. Forsyth in 1903.[2]

578. *C. Kramp.*—Arbogast's symbol for derivatives was adopted by C. Kramp, professor of mathematics and physics at Cologne, in his *Éléments d'Arithmétique universelle* (Cologne, 1808), pages 265,

D, signe des dérivations . . . . n.° 2.
$D.$, $D\cdot$, différence entre D sans point et avec point . . . . n.os 3 et 9.
$\underset{c}{D}^n$, $\underset{c}{D}^n.$, ... n.° 39.
$D^{m,n}$, $\underset{c}{D}^{m,n}$, $\underset{c}{D}^{l,m,n}.$, $\underset{c}{D}^{l,,}.\underset{c}{D}^{,m,}.\underset{c}{D}^{,,n}.$, ... n.° 111.
$\underset{c}{D}^m.\underset{c}{D}^{\prime n}.$, ... n.° 130.
${}^{\prime n}\underset{c}{D}^0.$, ${}^{\prime n}\underset{c}{D}^0.\underset{c}{D}^m.$, ... n.° 153.
$\underset{c}{D}^m.\underset{c}{D}^{\prime n}.\underset{c}{D}^{\prime\prime r}.$, $\underset{c}{D}^m.\underset{c}{D}^{\prime n}.\underset{c}{D}^{\prime\prime r}.\underset{c}{D}^{\prime\prime\prime s}.$, ... n.os 161 et 166.
$\mathcal{S}$, S^n, $S^{,1}$, $S^{1,}$, au lieu de D^{-1}, D^{-n}, $D^{,-1}$, $D^{-1,}$, ... n.os 36 et 391.
$D^n.1^{m:r.}$, $D^n.1^{m:-r.}$, ... n.° 422.
d, signe ordinaire de la différentiation, ... n.os 1 et 352.
$\underset{c}{d}^n$, ... n.° 352.
$d^{1,}$, $d^{,1}$, $d^{,,1}$, ... n.° 371.
$\int = d^{-1}$, signe ordinaire de l'intégration; $\int^n = d^{-n}$, ... n.° 384.
∂, $\underset{c}{\partial}^n$, $\partial.$, $\underset{c}{\partial}^n.$, ... n.° 352.
$\partial^{1,}$, $\partial^{,1}$, $\partial^{,,1}$, $\partial^{1,}.$, $\partial^{,1}.$, $\partial^{,,1}.$, ... n.° 376.
$\mathscr{f}$, $\mathscr{f}^n$, $\mathscr{f}.$, $\mathscr{f}^n.$, au lieu de ∂^{-1}, ∂^{-n}, $\partial^{-1}.$, $\partial^{-n}.$, ... n.° 384.
Δ, Σ, signes ordinaires des opérations qui donnent la différence et l'intégrale-somme.
$\Delta^{1,}$, $\Delta^{,1}$, ... n.° 409.
E, E^n, signes des états variés, ... n.° 442.
$E^{1,}$, $E^{,1}$, $E^{,,1}$, ... n.° 444.

FIG. 123.—From Arbogast's *Calcul des Dérivations* (1800), p. xxi

271. He takes $X = Ax^a + Bx^b + Cx^c +$, etc., and then designates by DX "ce que devient cette fonction, lorsqu'on multiplie tous ses termes par leurs exposans respectifs, et qu'ensuite on divise par x." Similarly for D^2X, etc. With great faith in the combinatorial analysis then flourishing in Germany under the leadership of Hindenburg,

[1] G. Tralles, *Abhandlungen Berliner Akademie*, Math. Klasse (1804–11), p. 214 ff.

[2] A. R. Forsyth, *Treatise on Differential Equations* (2d ed., 1888), p. 43; (3d ed., 1903), p. 55.

Kramp says (p. x): "Arbogast is obliged to assume the theorem of Taylor, as well as the ordinary operations of the differential calculus, as perfectly [*parfaitement*] known and demonstrated. My derivatives on the contrary are perfectly independent of all notion of the infinitely little or of the limit, of the differential or of differences; and is equally disinclined [*et bien loin*] to be established on the theorem of Taylor, this theorem does not appear but as a simple corollary of a proposition much more general." Under the heading "Notations" he refers to the notation dx and $dy:dx$ as follows: "Later researches have convinced me of the absolute inutility of this constant factor or divisor dx, as well as the notion of the infinitely small from which it has always been considered inseparable. In supposing it equal to unity one banishes all idea of the infinite and one causes all this part of analysis to re-enter the domain of ordinary algebra." The notation D is used in connection with the coefficients arising in his general polynomial development.

b) CRITICISMS OF EIGHTEENTH-CENTURY NOTATIONS

579. *R. Woodhouse.*—A critical comparison of the Newtonian and Leibnizian notations was made in 1803 by Robert Woodhouse,[1] of Cambridge in England. He states: "In the simplest cases, perhaps, there is not much exercise of choice, and $\dot{x}$, $\ddot{x}$, $\ddot{\ddot{x}}$, $\dot{x}^2$, $\dot{x}^3$, $\dot{x}^n$ are as neat as dx, d^2x, d^4x, dx^2, dx^3, dx^n. Take a case from Waring, p. 299, *Meditationes Analyticae*,

$$P\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^n}\right)+Q\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^{n-1}\,\dot{y}}\right)+\text{etc.; this by differential}$$

notation is, $P\cdot\dfrac{d^nV}{dx^n}+Q\cdot\dfrac{d^nV}{dx^{n-1}\,dy}+\text{etc.}$

in which it appears to me, that $\dot{x}^n$ is not so clearly distinguished from $\overset{n}{\dot{x}}$ as dx^n is from d^nx.

"Again, $\dddot{\overline{xy}}$ or $(xy)^{\cdots}$ is not so convenient as, $d^3(xy)$.

"Again, suppose x, x' x'', etc., to represent successive values of x, then according to the fluxionary notation, the first, second, etc., fluxions are $\dot{x}$, $\ddot{\overline{x'}}$, $\dddot{\overline{x''}}$; by differentials, dx, d^2x', d^3x''. Which notation has here the advantage, must be determined by inspection; and if the advantage is asserted to be with the fluxionary, it is impossible to state in words any irrefragable arguments to the contrary.

[1] R. Woodhouse, *Principles of Analytical Calculation* (Cambridge, 1803), p. xxvii.

"I put down a few more instances, $P\overset{r}{\dot{y}}+H\overset{r-1}{\dot{y}}+h\overset{r-2}{\dot{y}}$, (Waring, p. 206, *Meditat. Analyt.*) $Pd^ry+Hd^{r-1}y+hd^{r-2}y$; in the fluxionary notation the position of the symbols r, $r-1$, $r-2$, introduces ambiguity, since they may be mistaken for indices of P, H, h;

$$\left(\frac{\dot{x}}{\dot{z}}\right)^{\cdot} \qquad (fx)^{\cdot}, \qquad \frac{\dot{\dot{\dot{u}}}}{\dot{x}^2\,\dot{y}},$$

$$d\left(\frac{dx}{dy}\right) \qquad dfx, \qquad \frac{d^3u}{dx^2\,dy}, \ldots.$$

"The advantage on the side of the differential notation is not, it may be said, very manifest in these examples, and perhaps I had adhered to the notation most familiar to me, had not stronger reasons than what are contained in the preceding cases presented themselves, for adopting letters instead of dots as the significant symbols of operations. These reasons in a few words are: first, in the fluxionary notation, there is no simple mode of expressing the fluxionary or differential coefficients, that affect the terms of an expanded expression; thus, the form for the binomial without putting down the numeral coefficients, by fluxionary notation is,

$$(x+i)^m=x^m+\frac{\dot{\overline{x^m}}}{\dot{x}}\,i+\frac{\ddot{\overline{x^m}}}{1\cdot 2\dot{x}^2}\,i^2+\text{etc.},$$

an awkward mode of expression certainly; and even in the differential notation, the coefficients cannot be expressed, except by fractions, thus,

$$(x+i)^m=x^m+\frac{d(x^m)}{dx}\,i+\frac{d^2(x^m)}{1\cdot 2\cdot dx^2}\,i^2+\text{etc.},$$

but then, by a slight alteration of this notation, a very commodious one is obtained; thus, using the small capital D to denote the differential coefficient, that is, putting Dx^n for $\frac{d(x^n)}{dx}$, etc., we have,

$$(x+i)^m=x^m+Dx^m\cdot i+\frac{D^2x^m}{1\cdot 2}\cdot i^2+\text{etc.}\ldots\ldots$$

We have the elements of a very clear and symmetrical notation, when

Δ denotes the entire difference,

d the differential, or part of the entire difference,

D the differential coefficient,

δ the variation."

580. *S. F. Lacroix.*—Calculus notations are discussed by Lacroix,[1] who says in part: "Euler[2] stresses . . . . the defects of the English notation, which becomes difficult to write and also to perceive when the number of points exceeds three, and one does not see how it is possible to indicate it in figures without danger of confusion with exponents; moreover, we know that it is easy to omit a point in writing or that the point may fail to show in the print. Many of these objections apply to the notation of Lagrange, which has besides the great inconvenience of depriving analysts of the right to represent by the same letter, differently marked with accents, such quantities as have analogous signification. The use of d is not subject to these difficulties; this character is most conspicuous, especially if, considering it as a sign of operation, we write it in Roman type, and reserve the Italic letter for the designation of magnitudes, a practice which Euler did not follow: finally the numerals applied to the character d show very clearly the number of times the differentiation has been repeated." Lacroix states that eight symbols have been proposed for

[1] S. F. Lacroix, *Traité du Calcul différentiel et du Calcul intégral* (2d. ed.), Tome I (Paris, 1810), p. 246.

[2] L. Euler says in his *Institutiones calculi differentialis* (Petrograd, 1755), p. 100, 101: "This mode of symbolizing, as a matter of fact, cannot be disapproved, when the number of points is small, since it can be recognized instantly by counting; however, if many points are to be written, it carries with it the greatest confusion and very many inconveniences. The tenth differential or fluxion is represented in this exceedingly inconvenient manner $\overset{\vdots\vdots}{y}$, while by our mode of symbolizing, $d^{10}y$, it is most easily recognized. However, if a case arises in which much higher orders of differentials, perhaps even indefinite ones, are to be expressed, then the English notation is plainly unsuitable." On page 102 Euler says: "As in the differential calculus the letter d does not stand for a quantity, but for a sign [of operation], in order to avoid confusion in calculations, where many constant quantities occur, the letter d should not be employed for their designation; likewise one should avoid the introduction of the letter l as representing a quantity in calculation, when logarithms occur at the same time. It is to be desired that the letters d and l be written as slightly modified characters so as not to be confounded with the ordinary alphabetic forms which designate quantities. Just as, for example, in place of the letter r which first stood for radix [root], there has now passed into common usage this distorted form of it $\surd$." A concrete example of the confusion which may arise is given in *L'Enseignement mathématique*, Vol. I (Paris, 1899), p. 108, where Poincaré tells of an examination in mathematical physics in which the candidate became flustered and integrated the partial-differential equation $\frac{d^2z}{dt^2}=a^2\frac{d^2z}{dx^2}$, by dividing by d^2z, then multiplying by dx^2, thereby obtaining $\frac{dx^2}{dt^2}=a^2$ and $\frac{dx}{dt}=\pm a$.

the (total) first derivative, the $\frac{dy}{dx}$ of Leibniz, the $\frac{\dot{y}}{\dot{x}}$ of Newton, the $[x \perp y]$ of Landen, the $f(x)$ and $\frac{y'}{x'}$ of Lagrange, the $\varepsilon\, y$ of Pasquich, the $\frac{3y}{x}$ of Grüson, and the Dy of Arbogast. Lacroix takes the question of notations seriously (p. 248): "It is a principle avowed by all, that current notations should not be changed except when they are in manifest conflict with the concepts they are intended to represent, or when one can greatly shorten them, or finally when in modifying them, one renders them fit to develop new relations which cannot be exhibited without them. The signs of the differential calculus do not come under any of these cases: the enrichments which Mr. Lagrange has given to Analysis in his Théorie des fonctions, in his Résolution des équations numériques, in his Leçons sur le Calcul des fonctions may be represented with equal simplicity and elegance by the usual characters, as one can see in the first edition of this Treatise." Lacroix adds: "Before multiplying the signs in analysis that are already so very numerous that one may well stop to consider the embarrassment coming to those, who in studying and endeavouring to encompass the whole, must continually bring together formulas and operations that are analogous, yet are expressed in different characters." One should strive to "defend those [symbols] in which the Mécanique analytique and the Mécanique céleste are written."

581. *J. L. Lagrange.*—And what was Lagrange's attitude toward his own new symbolism?[1] In the autumn of his busy years his openmindedness is splendidly displayed in his attitude taken in the Preface to the second edition of his *Mécanique Analitique* (the spelling of the title in more recent editions is *Mécanique analytique*), where he retains the Leibnizian notation. He says:

"The ordinary notation of the differential calculus has been retained, because it answers to the system of the infinitely small, adopted in this treatise. When one has well comprehended the spirit of this system and has convinced himself of the exactitude of its results by the geometrical method of prime and ultimate ratios, or by the analytical method of derived functions, one may employ the infinitely little as an instrument that is safe and convenient for abbreviating and simplifying demonstrations. It is in this manner that one shortens the demonstrations of the ancients by the method of indivisibles."

[1] J. L. Lagrange, *Mécanique Analytique* in *Œuvres*, Tome XI (Paris), p. xiv.

Nor had Lagrange any prejudice against the Newtonian dots. He used them in his *Leçons sur le calcul des fonctions* (1806), page 442, for indicating "dérivées par rapport à $\dot{\imath}$," and in the second edition of his *Mécanique Analitique* to indicate time-derivatives. In the opinion of Thomson and Tait, "Lagrange has combined the two notations with admirable skill and taste."[1]

c) TOTAL DIFFERENTIATION DURING THE NINETEENTH CENTURY

582. *Barlow and Mitchell.*—A slight advantage of Lagrange's strokes over the Newtonian dots was that the former were not placed directly above the letter affected, but in the place where exponents are written. It is worthy of observation that at the opening of the nineteenth century a few English writers attempted to improve Newton's notation, in the case of fluxions of higher orders, by placing only one dot above the letter and then indicating the order by a number placed where exponents usually stand. Thus, Peter Barlow[2] writes $\dot{x}^n$ for the nth fluxion, but $\dot{x}^2$ for $(\dot{x})^2$. James Mitchell,[3] to denote the fluxions of a radical or fraction, places it in a parenthesis and writes the dot in the place for exponents. Sometimes, however, Mitchell and other authors write the fluxions of compound expressions by placing the letter F or f before them. This practice is, of course, objectionable, on account of the danger of misinterpreting the F or f as meaning, not "fluxion," but "function" or "fluent." We must remark also that this use of F or f was not new at this time; but is found much earlier in the writings of George Cheyne[4] in London (who also employed Φ to designate "fluxion of") and of A. Fontaine[5] in France.

583. *Herschel, Peacock, and Babbage.*—The adoption of the Leibnizian notation in England, advocated in 1802 by Woodhouse, was finally secured through the efforts of J. F. W. Herschel, Peacock, and Babbage at Cambridge. In his autobiography, Babbage says:[6]

[1] Thomson and Tait, *Treatise on Natural Philosophy* (new ed.; Cambridge, 1879), p. 158.

[2] P. Barlow, *Dictionary of Pure and Mixed Mathematics* (London, 1814), art. "Differential."

[3] James Mitchell, *Dictionary of the Mathematical and Physical Sciences* (London, 1823), art. "Fluxion."

[4] *Fluentium methodus inversa a Georgio Cheynaeo, M.D.* (Londini, 1703), p. 1 of the part against A. Pitcairne.

[5] Fontaine in *Histoire de l'académie r. des sciences*, année 1767 (Paris. 1770), p. 589 ff.

[6] Ch. Babbage, *Passages from the Life of a Philosopher* (1864), p. 39. Quoted in *Mathematical Gazette*, Vol. XIII, p. 163.

"The progress of the notation of Leibniz at Cambridge was slow. . . . . It is always difficult to think and reason in a new language, and this difficulty discouraged all but men of energetic minds. I saw, however, that, by making it [the tutors'] interest to do so, the change might be accomplished. I therefore proposed to make a large collection of examples of the differential and integral calculus. . . . . I foresaw that if such a publication existed, all those tutors who did not approve of the change of the Newtonian notation would yet, in order to save their own time and trouble, go to the collection . . . . to find problems to set. . . . . I communicated to Peacock and Herschel my view, and proposed that they should each contribute a portion [to the *Examples,* 1820]. . . . . In a very few years the change was completely established." Says Glaisher:[1]

"Peacock was Moderator in 1817, and he ventured to introduce the symbol of differentiation into the examination, his colleague, however, retaining the old fluxional notation. The old system made its appearance once more in 1818, but in 1819 Peacock was again Moderator, and, having a colleague who shared his views, the change was fully accomplished. The introduction of the notation and language of the differential calculus into the Senate House examination forms an important landmark in the history of Cambridge mathematics."

584. *A. L. Crelle.*—Influenced by Lagrange's *Théorie des fonctions analytiques,* Crelle bases his calculus[2] upon Taylor's theorem. His various notations may be displayed by the different ways of writing that theorem, as shown in Article 48, page 144:

"Die Anwendung der verschiedenen . . . . vorgeschlagenen Bezeichnungen auf die allgemeine Entwicklung von $f(X+k)$ würde in der Zusammenstellung folgende sein:

$$f(X+k)=|X+k|=fX+kdfX+\frac{k^2}{2}\,d^2fX \;\ldots\ldots\; +\frac{k^n}{1\cdot 2\ldots\ldots n}\,d^nfX$$

$$=|X|+kd|X|+\frac{k^2}{2}\,d^2|X| \;\ldots\ldots\; +\frac{k^n}{1\cdot 2\ldots\ldots n}\,d^n|X|, \text{ oder wenn } fX=Y$$

$$\text{heisst, } =Y+kdY+\frac{k^2}{2}\,d^2Y \;\ldots\ldots\; +\frac{k^n}{1\cdot 2\ldots\ldots n}\,d^nY. \text{ Ferner:}$$

$$f(X+k)=fX+DfX+\tfrac{1}{2}D^2fX \;\ldots\ldots\; +\frac{1}{1\cdot 2\ldots\ldots n}\,D^n|X|, \ldots\ldots \text{"}$$

[1] J. W. L. Glaisher's "Presidential Address," *Proceedings London Mathematical Society,* Vol. XVIII (London, 1886–87), p. 18.

[2] August Leopold Crelle, *Rechnung mit veränderlichen Grössen,* Band I (Göttingen, 1813).

The variables x, y, z are given in small capitals, which gives the page an odd appearance. It will be noticed that with Crelle d does not signify "differential"; it signifies *Abgeleitete Grösse*, the "derived function" of Lagrange, or the "derivation" of Arbogast. Crelle's notation in partial differentiation and in integration will be given later.

585. *A. L. Cauchy.*—That Cauchy should be influenced greatly by Lagrange is to be expected. If we examine his published lessons of 1823 on the infinitesimal calculus,[1] and his lessons of 1829 on the differential calculus, we find that he availed himself of the Leibnizian dx, dy, $\frac{dy}{dx}$ and also of the Lagrangian F' and y' for the first derivatives. In this respect he set a standard which has prevailed widely down to our own day.

586. *M. Ohm.*—Returning to Germany, we find another attempt to rebuild the calculus notations in the works of Martin Ohm.[2] His symbolism of 1829 is made plain by his mode of writing Taylor's theorem:

$$f(x+h)=f(x)+\partial f(x)\cdot h+\partial^2 f(x)\cdot\frac{h^2}{2!}+\ldots\ldots$$

One notices here the influence of Arbogast, though Ohm uses ∂ in place of D. Ohm writes also

$$\partial(A+B\cdot x^m)_x=mBx^{m-1}\ ,\quad \partial(a^x)_a=x\cdot a^{x-1}\ ,\quad \partial(a^x)_x=a^x\cdot\log a\ .$$

If y_x is a function of x, then $(\partial^\alpha y_x)_a$ is what becomes of $\partial^\alpha y_x$ when one writes a in place of x. He introduces also the Leibnizian dx, dy and writes

$$\frac{dy}{dx}=\partial y_x\ ,\qquad d^2y=\partial^2 y_x\cdot dx^2\ ,\qquad \frac{d^2y}{dx^2}=\partial^2 y_x\ .$$

Ohm's notation found some following in Germany. For instance, F. Wolff[3] in 1845 writes f_x to designate a function of x, and $y-y'=\partial f_{x'}(x-x')$ as the equation of a line through the point x', y'; Wolff uses also the differential notation dx, dy, d^2y, and the derivative $\frac{dx}{ds}$.

[1] Cauchy, *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal* (Paris, 1823); also his *Leçons sur le calcul différentiel* (Paris, 1829). Both are reprinted in his *Œuvres* (2d ser.), Vol. IV.

[2] Martin Ohm, *Versuch eines vollkommen consequenten Systems der Mathematik*, Vol. III (Berlin, 1829), p. 53, 67, 80, 94, 102, 103, 107.

[3] F. Wolff, *Lehrbuch der Geometrie*, Dritter Theil (Berlin, 1845), p. 178, 202, 278.

587. *Cauchy and Moigno.*—Meanwhile, some new designations introduced by Cauchy in France were clamoring for the right of way. Abbé Moigno[1] in the Introduction (p. xxi) to his calculus expresses regret that he could not altogether disregard custom and banish from his book "the vague and inconvenient notations $\frac{dy}{dx}$, $\frac{dy}{dx}dx$, $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{dz}{dx}dx$, $\frac{dz}{dy}\frac{dy}{dx}dx$, to substitute in their place the more compact notations $y'_x=D_xy$, d_xy, $z'_x=D_xz$, $z'_y=D_yz$, d_xz, d_yz, etc." On page 13 he lets d_xz be a differential and z'_x the derivative, where $z=F(y)$ and $y=f(x)$, so that the $d_xz=z'_xdx$ and therefore $\frac{d_xz}{dx}=\frac{d_yz}{dy}\cdot\frac{d_xy}{dx}$, but "it is the custom to suppress" the indices x and y and to write $\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$.

588. *B. Peirce.*—Arbogast's D for derivative was adopted by Benjamin Peirce[2] in his *Curves, Functions, and Forces*, the first volume of which was published in Massachusetts in 1841. The calculus notation used in the United States before 1824 was almost exclusively Newtonian, from 1824 to 1841 almost entirely Leibnizian. In B. Peirce's book, "Differential coefficients are denoted by D, D', etc.; thus $Df\cdot x=\frac{df\cdot x}{dx}$, $D^2\cdot f\cdot x=D\cdot D\cdot f\cdot x$, or, since dx is independent of x, $D^2\cdot f\cdot x=\frac{d\cdot Df\cdot x}{dx}=\frac{d^2f\cdot x}{dx^2}$." Again, "differentials are denoted by the letters δ, δ' etc., d, d', etc."; Peirce writes $e^x=D^n_\cdot e^x$, and $D\ \text{tang}^{[-1]}x=Dy=\frac{dy}{dx}=(1+x^2)^{-1}$. In Volume II (1846), page 15, he lets "also $d^n_c\cdot f\cdot x_0$ be the first differential coefficient of $f\cdot x_0$ which does not vanish," where $f\cdot x_0\mp 0$ or ∞. Here the d_c meant "differential coefficient."

Since the time of B. Peirce the D has held its place in a few of America's best texts, but even in those it has not been used to the complete exclusion of the Leibnizian symbols. Some authors introduce the Dy, in preference to the $\frac{dy}{dx}$, to guard the student against the misconception of regarding the derivative as a fraction. Later in

[1] L'Abbé Moigno, *Leçons de calcul Différentiel et de calcul Intégral, rédigées d'après les méthodes et les ouvrages publiés ou inédits de M. A.-L. Cauchy,* Tome I (Paris, 1840).

[2] Benjamin Peirce in his *Elementary Treatise on Curves, Functions, and Forces* (new ed.; Boston and Cambridge), Vol. I (1852), p. 179, 182, 193, 203. (First edition, 1841).

the book they begin to avail themselves of the flexibility afforded by the $\frac{dy}{dx}$, which allows an easy passage from derivative to differential, or differential to derivative, by the application of simple algebraic rules.

589. *G. S. Carr*[1] introduced "experimentally" a radical departure from the usual notations; he lets y_x stand for $\frac{dy}{dx}$, y_{2x} for $\frac{d^2y}{dx^2}$, y_{3x} for $\frac{d^3y}{dx^3}$, and so on. This "shorter notation" has not met with general favor, for obvious reasons.

590. The more minute examination of the continuity of functions gave rise to the consideration of right-hand and of left-hand derivatives and to corresponding notations. Peano[2] uses D for derivative, writing $D(f, u, x)$ for the derivative of f in the class u, for the value x of the variable, and $D(f, x+Q_0, x)$ for the *dérivée à droite*, and $D(f, x-Q_0, x)$ for the *dérivée à gauche*.

591. *G. Peacock*.—A curious extension of the notion of a derivative $\frac{d^ny}{dx^n}$ to the case where n is not a positive integer, say $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}$, was considered by Leibniz (§ 532), Euler, Fourier, Cauchy, and Liouville. The notion was first applied to differentials. Euler[3] wrote $\frac{d^{\frac{1}{2}}x}{\sqrt{dx}}=2\sqrt{\frac{x}{\pi}}$, while according to the definition of Leibniz one has $\frac{d^{\frac{1}{2}}x}{\sqrt{dx}}=\sqrt{x}$. The definition of Liouville[4] agrees with that of Leibniz. The subject received the particular attention of George Peacock,[5] for the purpose of illustrating his "principle of the permanence of equivalent forms." Following Liouville, Peacock takes, for example,

$$\frac{d^n(e^{mx})}{dx^n}=m^ne^{mx}$$

[1] G. S. Carr, *Synopsis .... of Pure Mathematics* (London, 1886), p. 258, § 1405.

[2] G. Peano, *Formulaire mathématique*, Tome IV (Turin, 1903), p. 168.

[3] L. Euler, *Comment. Academiae Petropol. ad annos 1730 et 1731*, Vol. V, p. 36–57. See M. Cantor, *Vorlesungen*, Vol. III (2d ed.), p. 655.

[4] See G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XIV (1913–14), p. 80, 81; C. W. Borchardt, *Monatsberichte d. Akad. d. Wissensch. zu Berlin* (1868), p. 623–25.

[5] George Peacock, "Report on the Recent Progress and Present State of Certain Branches of Analysis," *British Association Report for 1833* (London, 1834), p. 211–21, 241–49. Peacock gives Bibliography.

and then considers the propriety of assuming this relation as a definition of the operation when n is not necessarily a positive integer. This speculation did not lead to results commensurate with those obtained from the generalization of n in the exponential notation a^n, but it illustrates how a suitable notation is capable of suggesting generalizations which a purely rhetorical exposition would not readily do. In England this generalized differentiation demanded the attention also of D. F. Gregory, P. Kelland, and especially of Oliver Heaviside who made important use of it in electromagnetic theory.[1]

592. *J. Fourier.*—Fourier, in his *Traité de la chaleur* (1822), showed in connection with the solution of partial differential equations the advantage which may accrue from the separation of the symbols of operation from those of quantity. Thus, in chapter ix, Section IV: "If the proposed differential equation is

$$\frac{d^2v}{dt^2}+\frac{d^4v}{dx^4}+2\frac{d^4v}{dx^2dy^2}+\frac{d^4v}{dy^4}=0\ ,$$

we may denote by D_Φ the function $\frac{d^2\Phi}{dx^2}+\frac{d^2\Phi}{dy^2}$, so that DD_Φ or D^2_Φ can be formed by raising the binomial $\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)$ to the second degree, and regarding the exponents as orders of differentiation."

This symbolic method was developed further in England by Greatheed[2] and D. F. Gregory,[3] both of Cambridge; by Boole,[4] of Queen's University, Ireland; Carmichael[5] of Dublin; and by many others in Great Britain and on the Continent.

d) PARTIAL DIFFERENTIALS AND PARTIAL DERIVATIVES

593. *L. Euler.*—Partial derivatives appear in the writings of Newton, Leibniz, and the Bernoullis, but as a rule without any special symbolism. To be sure, in 1694, Leibniz,[6] in a letter to De l'Hospital, wrote "δm" for the partial derivative $\frac{\partial m}{\partial x}$ and "ϑm" for $\frac{\partial m}{\partial y}$, and De

[1] *Prcceedings of the Royal Society of London*, Part I (Feb. 2, 1893); Part II (June 15, 1893); Part III (June 21, 1894). See also O. Heaviside, *Electromagnetic Theory*, Vol. II (London, 1899), p. 434, 435.

[2] S. S. Greatheed in *Philosophical Magazine* (Sept., 1837).

[3] D. F. Gregory in *Cambridge Mathematical Journal*, Vol. I, p. 123.

[4] George Boole, *Differential Equations* (4th ed.; London, 1877), chaps. xvi, xvii.

[5] Robert Carmichael, *Calculus of Operations* (London, 1855).

[6] *Leibnizens Mathematische Schriften*, Vol. II (Berlin, 1850), p. 261, 270.

l'Hospital uses "δm" in his reply of March 2, 1695 (see § 532). In 1728 L. Euler[1] used new letters, P, Q, R, to designate the partial derivatives with respect to x, y, z, respectively. Similarly, G. Monge[2] used the small letters p and q as partial derivatives of z with respect to x and y, respectively, and r, s, t as partial derivatives of the second order. In more recent time Monge's abbreviations have been employed by writers on differential equations and differential geometry. Euler indicated in 1755, in his *Institutiones calculi differentialis*, Volume I, page 195, the partial derivative of P with respect to y by the use of a parenthesis. Thus, $\left(\frac{dP}{dy}\right)$ signifies that "P is so differentiated that y alone is treated as a variable, and that differential is divided by dy."[3]

The notation $\left(\frac{dy}{dx}\right)$ for partial derivatives has been widely used, notwithstanding the objections which have been raised to it. If one writes

$$\left(\frac{dp}{dx}\right)^2+\left(\frac{dp}{dy}\right)^2+\left(\frac{dp}{dz}\right)^2=1 \text{ or } a\left(\frac{dy}{dx}\right) ,$$

the reader may be in doubt whether the parentheses are introduced to indicate the presence of partial derivatives or merely to render algebraic service in the expression of the square or the product. These objections to the contrary notwithstanding, this Eulerian notation long held its place in competition with other notations for partial derivatives. Some idea of its wide distribution both in time and localities will be conveyed by the following references: It was used by L'Abbé Sauri,[4] of Paris; Gherli,[5] in Modena; Waring,[6] of Cam-

[1] *Commentarii Academiae Scientiarum imperialis, Petropolitanae*, Tome III (1728) (ed. 1732), p. 115, 116; *Encyclopédie des sciences mathématiques*, Tome II, Vol. I (Paris and Leipzig, 1912), p. 284, n. 168.

[2] Gaspard Monge, *Application de l'analyse à la géométrie* (edited after 4th ed. of 1809 by J. Liouville; Paris, 1850), p. 53 and 71.

[3] "Brevitatis gratia autem hoc saltem capite quantitates r et q ita commode denotari solent, ut r indicetur per $\left(\frac{dP}{dy}\right)$, quae scriptura designatur P ita differentiari, ut sola y tanquam variabilis tractetur, atque differentiale istud per dy dividatur." Just above that passage Euler wrote "posito ergo x constante erit $dP=rdy$. . . . . Deinde posito y constante erit $dQ=qdx$."

[4] L'Abbé Sauri, *Cours Complet de mathématiques*, Tome V (Paris, 1774), p. 83.

[5] O. Gherli, *Gli Elementi teorico-pratici delle matematiche pure*, Tomo VII (Modena, 1777), p. 2.

[6] E. Waring, *Meditationes analyticae* (Cambridge, 1785), p. 166.

bridge; Laplace,[1] in Paris; by Monge,[2] Legendre,[3] Lagrange,[4] in 1759, 1792; by De Montferrier,[5] in Paris; Price,[6] at Oxford; Strong,[7] in New York; Resal,[8] at Paris; Newcomb,[9] at Washington; and Perry,[10] of London, who in 1902 expressed his preference for the Eulerian notation.

In 1776 Euler[11] uses $\frac{\partial^\lambda}{p} \cdot V$ to indicate the λth derivative, partial with respect to the variable p; he uses a corresponding notation $\frac{S}{x} \cdot V$ for partial integration. A similar notation is adopted by F. Servois[12] in 1814.

594. *W. J. G. Karsten.*—The need of some special designation must have been felt by many workers. Apparently unaware of Euler's notation, Karsten[13] in 1760, taking V as a function of x, y, z, u, etc., denotes the partial increment with respect to x by $\overset{x}{\Delta} V$, the partial increment of $\overset{x}{\Delta} V$ with respect to y by $\overset{y\,x}{\Delta\Delta} V$; correspondingly, he writes the partial differentials $\overset{x}{d} V$ and $\overset{y\,x}{dd} V$; he designates also the partial

[1] P. S. Laplace, *Traité de mécanique céleste*, Tome I (Paris), Vol. VII (1798), p. 7. *Histoire de l'académie r. des sciences*, année 1772 (Paris, 1775), *Mémoires*, p. 346.

[2] G. Monge in *Histoire de l'académie r. des sciences*, année 1784 (Paris, 1787), p. 119.

[3] A. M. Legendre, in *op. cit.*, année 1784, p. 372.

[4] J. L. Lagrange in *Mémoires de l'académie r. des sciences et belles-lettres*, année 1792 et 1793 (Berlin, 1798), p. 302; *Miscellanea philosophico-mathematica societatis privatae Taurinensis*, Tome I, "Propagation du son," p. 18.

[5] A. S. de Montferrier, *Dictionnaire des sciences mathématiques*, Tome I (Paris, 1835), art. "Différence."

[6] B. Price, *Treatise on Infinitesimal Calculus*, Vol. I (Oxford, 1852), p. 118.

[7] Theodore Strong, *Treatise of the Differential and Integral Calculus* (New York, 1869), p. 44.

[8] H. Resal, *Traité élémentaire de mécanique céleste* (2d ed.; Paris, 1884), p. 8.

[9] Simon Newcomb, *Elements of the Differential and Integral Calculus* (New York, 1887), p. 54.

[10] John Perry in *Nature*, Vol. LXVI (1902), p. 271.

[11] L. Euler in *Nova acta academiae i. scientiarum Petropolitanae*, Tome IV, ad annum 1786 (Petrograd, 1789), p. 18.

[12] F. Servois in *Gergonne's Annales de mathématiques*, Tome V (Nismes, 1814, 1815), p. 94, 95.

[13] W. J. G. Karsten, *Mathesis Theoretica Elementaris atque sublimior* (Rostochii, 1760), p. 775, 781, 782, 807, 808.

differential with respect to y of the partial integral with respect to x, $\overset{y}{d}\overset{x}{\int} Pdx$. Similarly, for partial derivatives he writes $\frac{\overset{y}{d}P}{dy}=\frac{\overset{x}{d}Q}{dx}$.

This notation did not attract much attention. It involved, in the last illustration, four terraces of type, which was objectionable. Moreover, German writers in Karsten's time were not in the limelight, and not likely to secure a following outside of their own country.

595. *A. Fontaine.*—Attention must be paid to the notation in partial differentiation used by Alexis Fontaine,[1] of Paris, who presented certain memoirs before the French Academy in 1739, but which were not published until 1764. The notation for partial differentials used by him is found in textbooks even of our own time.

In the Table of Contents of the *Memoirs* contained in the volume, one finds μ given as a function of x, y, z, u, etc., and then

$$d\mu=\frac{d\mu}{dx}dx+\frac{d\mu}{dy}dy+\frac{d\mu}{dz}dz+\frac{d\mu}{du}du+\text{etc}\,.$$

$$\frac{dd\mu}{dx}=\frac{dd\mu}{dx^2}dx+\frac{dd\mu}{dx\,dy}dy+\frac{dd\mu}{dx\,dz}dz+\frac{dd\mu}{dx\,du}du+\text{etc.}\,,$$

and similarly expressions for $\frac{dd\mu}{dy}$, $\frac{dd\mu}{dz}$, $\frac{dd\mu}{du}$. Evidently, the $d\mu$ in the left member of the first equation is the total differential of the first order, the $\frac{dd\mu}{dx}$ in the left member of the second equation is the total differential of the partial derivative $\frac{d\mu}{dx}$. The partial differential with x as the sole variable is here indicated by $\frac{du}{dx}\,dx$; with y as the sole variable, by $\frac{d\mu}{dy}\,dy$; and similarly for z, u, etc. This notation lacks compactness, but has the great merit of being easily understood and remembered.

Fontaine's equations in the Table of Contents are followed by the following explanation:

"Cette expression-ci $\frac{1}{dx}\cdot d\mu$ est donc bien différente de celle-cu $\frac{d\mu}{dx}$.

[1] *Mémoires donnés à l'académie royale des sciences, non imprimés dans leur temps.* Par M. Fontaine, de cette académie (Paris, 1764). The part in the book containing definitions, notations, and fundamental theorems of the *calcul intégra* on p. 24 is dated Nov. 19, 1738.

La première signifie la différence de μ divisée par dx; la seconde signifie le coëfficient de dx dans la différence de μ."

In other words, here the $\frac{d\mu}{dx}$ is our modern $\frac{\partial\mu}{\partial x}$. As the only derivatives occurring in Fontaine's equations are partial derivatives, no confusion could arise in the interpretation of derivatives; $\frac{dd\mu}{dx\,dy}\,dy$ meant the partial differential (with respect to y as the only variable) of the partial derivative $\frac{\partial\mu}{dx}$, while $\frac{dd\mu}{dx\,dy}$ indicated $\frac{\partial^2\mu}{\partial x\,\partial y}$. This interpretation is confirmed by statements on page 25.

The wide spread of Fontaine's notation for partial differentials may be displayed as follows: It was used by D'Alembert,[1] Lagrange,[2] Poisson,[3] Gauss,[4] Moigno,[5] Duhamel,[6] Hermite,[7] and Davies and Peck.[8]

A notation more along the line of Karsten's is found in Abbé Girault[9] de Koudou's calculus, which is a curious mixture of English phraseology and German symbols. We quote: "Cette expression $\overset{x}{\overline{d(X)}}$ signifie Fluxion de X dans la supposition de x seule variable," where $X=x+y+z-u+a$.

596. *Monge, Condorcet, and Legendre.*—About 1770 G. Monge[10] wrote $\frac{\delta V}{dx}$ and $\frac{dV}{dy}$ to represent the partial derivatives with respect to

[1] D'Alembert in *Mémoires de l'académie*, Tome XIX, année 1763 (Berlin, 1763), p. 264.

[2] J. L. Lagrange, *Méchanique analitique* (1788), p. 137.

[3] S. D. Poisson in *Mémoires de l'académie r. des sciences de l'Institut de France*, année 1816, Tome I (Paris, 1818), p. 21.

[4] Gauss, *Werke*, Vol. V (Göttingen, 1867), p. 57. Gauss says: "Spectari itaque potererit z tamquam functio indeterminatarum x, y, cuius differentialia partialia secundum morem suetum, sed omissis vinculis, per $\frac{dz}{dx}\cdot dx$, $\frac{dz}{dy}\cdot dy$ denotabimus."

[5] L'Abbé Moigno, *Leçons de calcul différentiel et de calcul intégral* (Paris, 1840), Tome I, p. 20.

[6] J. M. C. Duhamel, *Éléments de calcul infinitésimal* (3d ed.; Paris, 1874), p. 273.

[7] Ch. Hermite, *Cours d'analyse*, 1. Partie (Paris, 1873), p. 80.

[8] Charles Davies and William G. Peck, *Mathematical Dictionary* (New York and Chicago, 1876), art. "Partial Differential."

[9] *Leçons analytiques du Calcul des Fluxions et des Fluentes, ou Calcul Differentiel et Intégrale.* Par M. L'Abbé Girault de Koudou (Paris, MDCCLXXII).

[10] Monge in *Miscellanea Taurinensia*, Tome V, Part II (1770–73), p. 94.

x and y, respectively. In an article of the year 1770 Marquis de Condorcet[1] indicated a partial differential of z with respect to x by dz, and a partial differential of z with respect to y by ∂z, or else he indicated by dz a total differential and by ∂z a partial. "Jusqu'ici l'une ou l'autre hypothèse donne le même résultat, mais lorsqu'il en sera autrement, j'aurai soin d'avertir de celle dont il sera question." In an article[2] of 1772 he alters the meaning of d and ∂ so as to signify differentiation, with respect to y and x, respectively.

The use of the rounded letter ∂ in the notation for partial differentiation occurs again in 1786 in an article by A. M. Legendre.[3] He introduces the modern notation $\frac{\partial v}{\partial x}$ for the partial derivative and says: "Pour éviter toute ambiguité, je représenterai par $\frac{\partial v}{\partial x}$ le coéfficient de dx dans la différence de v et par $\frac{dv}{dx}$ la différence complète de v divisée par dx."

How did it happen that Legendre's happy suggestion remained unheeded? There were perhaps two reasons. One was that he himself failed to use his notation in later articles. The year following he not only abandoned his own notation, but also that of Euler which he had used in 1784, and employed no symbolism whatever to distinguish between total and partial differentiation. Perhaps a second reason lies in the fact that at that time there was a great scramble for the use of the letters D and d in different fields of mathematics. The straight d had been used by Leibniz and his followers, but writers on finite differences[4] laid claim to it, as well as to the capital D.

Thereupon Arbogast gave D the new meaning of "derivation." It looked indeed as if the different mathematical architects engaged in erecting a proud mathematical structure found themselves con-

[1] Condorcet in *Histoire de l'académie r. des sciences*, année 1770 (Paris, 1773), p. 152.

[2] Condorcet in *op. cit.*, année 1772, Part I (Paris, 1775), *Mémoires*, p. 14.

[3] Legendre in *op. cit.*, année 1786 (Paris, 1788), p. 8. The article is translated by Stäckel in *Ostwald's Klassiker der exakten Wissenschaften*, No. 78 (Leipzig, 1896), p. 64–65. See also *Encyclopédie des sciences mathématiques*, Tome II, Vol. I, p. 284, n. 168.

[4] Among the writers who used the D in finite differences were L'Abbé Sauri, *Cours complet de mathématiques*, Tome III (Paris, 1774), p. 304; Georg Vega, *Vorlesungen über die Mathematik*, Vol. II (Wien, 1784), p. 395; J. A. da Cunha, *Principios mathematicos* (1790), p. 269. The small letter d was used less frequently in finite differences; we cite Charles Hutton who in his *Mathematical and Philosophical Dictionary* (London, 1895), art. "Differential Method," employs the d.

fronted with the curse of having their sign language confounded so that they could the less readily understand each other's speech. At this juncture certain writers on the calculus concluded that the interests of their science could be best promoted by discarding the straight-letter d and introducing the rounded ∂. Accordingly they wrote ∂y for the total differential, $\frac{\partial y}{\partial x}$ for the total derivative, and $\left(\frac{\partial y}{\partial x}\right)$ for the partial derivative. G. S. Klügel refers to this movement when he says in 1803:[1] "It is necessary to distinguish between the symbol for differential and that for a [finite] quantity by a special form of the letter. In France[2] and in the more recent memoirs of the Petersburg Academy, writers[3] have begun to designate the differential by the curved ∂." So Klügel himself adopts this symbolism from now on for his dictionary. Euler's *Institutiones calculi integralis*, which in its first edition of 1768–70 used the straight d, appeared in the third edition of 1824 with the round ∂, both for the total differential and the total derivative. This same notation is found in J. A. Grunert's calculus[4] of 1837. However, the movement failed of general adoption; for some years both d and ∂ were used by different writers for total differentiation.

In view of these countercurrents it is not surprising that Legendre's genial notation for partial differentiation failed of adoption.

597. *J. L. Lagrange.*—Lagrange, in his *Méchanique analitique* of 1788, used no special signs for partial differentiation; he even discarded Euler's parentheses. The reader was expected to tell from the context whether a derivative was total or partial. We have seen that this attitude was taken also by Legendre in 1787.

598. *S. F. Lacroix.*—Lagrange's procedure of 1788 was adopted by Lacroix whose three-volume treatise on the calculus was a standard work of reference in the early part of the nineteenth century. Lacroix says:[5] "First, let me observe that it [the partial derivative]

[1] G. S. Klügel, *Mathematisches Wörterbuch*, 1. Abth., 1. Theil (Leipzig, 1803), art. "Differentiale."

[2] In France, Le Marquis de Condorcet had used the ∂ in writing total derivatives in his *Probabilité des décisions*, in 1785.

[3] The *Nova Acta Petropolitana*, the first volume of which is for the year 1785, contain the rounded ∂.

[4] Johann August Grunert, *Elemente der Differential und Integralrechnung* (Leipzig, 1837).

[5] S. F. Lacroix, *Traité du calcul différentiel et du calcul integral*, Tome I (2d ed.; Paris, 1810), p. 243.

should be disencumbered of the parentheses which Euler employed. Really, $\frac{dz}{dx}$, $\frac{dz}{dy}$ are as clear as $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ when one knows beforehand that z is a function of two independent variables x and y, which the statement of the question, or the meaning of which it is susceptible, always indicates. . . . . Messrs. Lagrange and Legendre have long ago suppressed the parentheses in their calculations and I have felt that, without inconvenience, I may follow the example which they have set. Fontaine who was the first to apply the notation of Leibniz to partial differentials, proposed to designate by $\frac{1}{dx}\,d\mu$ the ratio of dx to the total differential of $d\mu$." In a more complicated function of several variables, of say x, y, z, where perhaps z is a function of x and y, Lacroix admits that some special notations should be used such as had been proposed by Lagrange (in 1797 and 1801).

Lacroix' attitude toward partial derivatives has had a very large following everywhere, almost down to the present; no special notation was employed for the ordinary partial derivative. This was the attitude of Poisson,[1] Cauchy (in 1821 and 1822),[2] Serret,[3] Duhamel,[4] Hermite,[5] and others in France. This was the practice of Boole,[6] Todhunter,[7] Lord Rayleigh,[8] Thomson and Tait,[9] in Great Britain, it being remarked by W. H. Young,[10] as late as 1910, that in English writings the ordinary $\frac{df}{dx}$ was sometimes used to represent the partial derivative. In Sweden a similar course was followed, as is exemplified in papers of A. V. Bäcklund.[11]

[1] S. P. Poisson, *Traité de mécanique* (2d ed.), Tome I (Paris, 1833), p. 36.

[2] A. Cauchy, *Mémoires de l'académie r. des sciences*, Tome V, années 1821 et 1822 (Paris, 1826), p. 47.

[3] *Notes de M. J. A. Serret sur le calcul différentiel et le calcul intégral par Lacroix* (no date, but after 1861). But in J. A. Serret's *Cours de calcul différentiel et intégral* (2d ed.; Paris, 1879), p. 29, etc., the Legendre-Jacobi notation is used.

[4] J. M. C. Duhamel, *Éléments de calcul infinitésimal* (3d ed.; Paris, 1874), p. 252.

[5] Ch. Hermite, *Cours d'Analyse*, 1. Partie (Paris, 1873), p. 78, 79.

[6] George Boole, *Differential Equations.*

[7] I. Todhunter, *Treatise on the Differential Calculus* (7th ed.; London, 1875)

[8] Baron Rayleigh, *The Theory of Sound* (London, 1877).

[9] Thomson and Tait, *Treatise on Natural Philosophy* (1879), p. 160.

[10] W. H. Young, *Fundamental Theorems of the Differential Calculus* (Cambridge University Press, 1910), p. 20.

[11] A. V. Bäcklund in *Lunds Univ. Årsskrift*, Tom. IX (Lund. 1872), p. 4.

This failure to observe a notational distinction between partial and total derivatives caused the less confusion, because the derivatives were usually obtained from expressions involving differentials, and in the case of differentials some distinctive symbolism was employed—it being either Fontaine's designation or some later notation.

599. *J. A. da Cunha.*—Portugal possessed an active mathematician, Da Cunha,[1] who in 1790, in a wonderfully compact, one-volume treatise of mathematics, used symbolism freely. With him, "d^x denotes the fluxion taken with respect to x, $\int^x$ the fluent taken relative to x." When $M=\frac{d^yO}{dy}$, then $\frac{d^xM}{dx}=\frac{`d^2O}{dy\,dx`}$, the single, inverted commas in the second derivative being used as a reminder that each of the two differentiations is partial.

600. *S. Lhuilier.*—A notation for partial differentials different from Fontaine's and resembling Karsten's is given in Simon Lhuilier's book on the exposition of the principles of the calculus,[2] in 1795. When P is a function of x and y, he writes the first three partial differentials with respect to x thus: ${}^xd'P$, ${}^xd''P$, ${}^xd'''P$; in general, he indicates N partial differentiations with respect to x, followed by M partial differentiations with respect to y, by the differential symbol ${}^yd^M\,{}^xd^NP$. We shall see that Cauchy's notation of 1823 differs from Lhuilier's simply in the positions of the y and x and in the use of small letters in the place of M and N.

601. *J. L. Lagrange.*—We return to the great central figure in the field of pure mathematics at the close of the eighteenth century—Lagrange. The new foundations which he endeavored to establish for the calculus in his *Théorie des fonctions analytiques* (1797) carried with it also new notations, for partial differentiation. He writes (p. 92) f', f'', f''', ... , to designate the first, second, third, etc., functions (derivatives) of $f(x, y)$, relative to x alone; and the symbols f_1, f_{11}, f_{111}, ... , to designate the first, second, third, ... , derivatives, relative to y alone. Accordingly, he writes (p. 93) also $f_1'(x, y)$, $f_1''(x, y)$, $f_{11}'(x, y)$, $f_{11}''(x, y)$, etc., for our modern $\frac{\partial^2 f}{\partial y\,\partial x}$, $\frac{\partial^3 f}{\partial y\,\partial x^2}$, $\frac{\partial^3 f}{\partial y^2\,\partial x}$, $\frac{\partial^4 f}{\partial x^2\partial y^2}$, etc. In the case of a function $F(x, y, z)$ of three variables he

[1] J. A. da Cunha, *Principios mathematicos* (Lisboa, 1790), p. 263.

[2] S. Lhuilier, *Principiorum calculi differentialis et integralis expositio elementaris* (Tübingen, 1795), p. 238.

represents (*Œuvres*, Vol. IX, p. 158), by $F'(x)$, $F'(y)$, $F'(z)$ the *primes fonctions* of $F(x, y, z)$ taken with respect to x, y, z, respectively, as independent variables. The danger of this notation is that $F'(x)$ might be taken to be the primitive F' function of the single variable x. Lagrange also (*ibid.*, p. 177) denotes by u', u_1, and ${}_1u$ the prime functions (first derivatives) of u with respect to x, y, z, respectively. Furthermore (*ibid.*, p. 273), u'', u_{11}, u_1' stand for our $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\,\partial y}$, respectively. In 1841 C. G. J. Jacobi declared that "the notation of Lagrange fails, if in a function of three or more variables one attempts to write down differentials of higher than the first order."[1] But Stäckel[2] remarked in 1896 that the Lagrangian defects can be readily removed by the use of several indices, as, for example, by writing $f_{\alpha\beta\gamma}$ for our $\frac{\partial^{\alpha+\beta+\gamma}f}{\partial x^\alpha \partial y^\beta \partial z^\gamma}$.

The historical data which we have gathered show that as early as 1760 Karsten had evolved a precise notation even for higher partial derivatives. It was capable of improvement by a more compact placing of the symbols. But this notation remained unnoticed.

That Lagrange was not altogether satisfied with his notation of 1797 appears from his *Résolution des équations numériques* (1798), page 210, where he introduces $\left(\frac{Z'}{a'}\right), \left(\frac{Z'}{b'}\right), \left(\frac{Z''}{a'^2}\right), \left(\frac{Z''}{a'b'}\right)$ to designate the partial derivatives $\frac{\partial Z}{\partial a}, \frac{\partial Z}{\partial b}, \frac{\partial^2 Z}{\partial a^2}, \frac{\partial^2 Z}{\partial a\,\partial b}$, and then says: "Cette notation est plus nette et plus expressive que celle que j'ai employée dans la *Théorie des fonctions*, en plaçant les accens différemment, suivant les différentes variables auxquelles ils se rapportent." Lagrange adds the further comment: "In substituting the former in place of this, the algorithm of derived functions conserved all the advantages of the differential calculus, and will have this additional one of disencumbering the formulas of that multitude of d's which lengthen and disfigure them to a considerable extent, and which continually recall to mind the false notion of the infinitely little."

It will be noticed that Lagrange's notation for partial derivatives, as given in 1798 in his *Résolution des Équations numérique*, and again in his *Leçons sur le calcul des fonctions* (Paris, 1806), page 347,[3] closely

[1] C. G. J. Jacobi, "De determinantibus functionalibus," *Journal für d. r. & a. Mathematik*, Vol. XXII, p. 321; Jacobi, *Ges. Werke*, Vol. III, p. 397.

[2] P. Stäckel in *Ostwald's Klassiker der exakten Wiss.*, No. 78, p. 65.

[3] J. Lagrange, *Œuvres*, Vol. X.

resembles the notation previously used by Waring at Cambridge. The only difference is that Waring employs the fluxional dot, placed above the letter, as in $\left(\frac{\dot{Z}}{\dot{a}}\right)$, while Lagrange uses a stroke in the position assigned to an exponent, as in $\left(\frac{Z'}{a'}\right)$. It will not escape the reader that in ordinary derivatives as well, the Lagrangian f', f'', f''' bears close resemblance to the Newtonian $\dot{x}$, $\ddot{x}$, $\dddot{x}$, the stroke again replacing the dot. We shall point out that certain English and American writers shifted the Newtonian dots to the position where exponents are placed (§§ 579, 630). By so doing the notations of Newton and Lagrange were brought still closer together.

602. *L. F. A. Arbogast.*—In his *Calcul des Dérivations* (1800), page 89, Arbogast designates by $D^{m}\cdot\varphi(a, \alpha)$ m total derivations, while $D^{m,n}\cdot\varphi(a, \alpha)$ designates partial derivations, viz., m derivations with respect to a only, followed by n derivations with respect to α only. Accordingly, $D^{o,n}\cdot\varphi(a, \alpha)$ or simply $D^{,n}\varphi(a, \alpha)$ indicates n partial derivations with respect to α. Also $\underset{c}{D^{m,n}}\cdot\varphi(a, \alpha)$ means that $D^{m,n}\cdot\varphi(a, \alpha)$ is to be divided by $(m!\, n!)$. He uses the Leibnizian d as the sign for the ordinary differential. The symbols $d^{\prime ,}$, $d^{,\prime}$, are partial differentials (p. 319). He writes $d^{,\prime}+d^{\prime ,}$, $d^{n}=(d^{,\prime}+d^{\prime ,})^{n}$, $d^{n}(x, y, z)=(d^{,,\prime}+d^{,\prime}+d^{\prime ,})^{n}\times xyz$.

603. *J. L. Lagrange.*—In close alignment with the Arbogast symbols is the new notation for partial derivatives which Lagrange gave in his *Leçons sur le calcul des fonctions* (Paris, 1801; 2d ed., 1806), pages 329, 330, 331. Let $f(x, y)$ be a function of x and y, then $f^{\prime ,}(x, y)$ means the partial derivative with respect to x, $f^{,\prime}(x, y)$ the partial derivative with respect to y, $f^{\prime ,\prime}(x, y)$ the partial derivative with respect to x, followed by the partial derivative with respect to y. Higher partial derivatives are designated by $f^{m,n}(x, y)$.

Robert Woodhouse referred to these signs in his *Principles of Analytical Calculation* (Cambridge, 1803), page xxx, where he compares the English and continental notations and gives his reasons for adopting the continental. He says: "Many theorems are conveniently demonstrated by separating the scale of operation, thus, when applied to a rectangle xy, Δ, D, d, may be separated into parts, as $\Delta^{\prime ,}$, $\Delta^{,\prime}$, $D^{\prime ,}$, $D^{,d}$, $d^{\prime ,}$, $d^{,\prime}$, where $\Delta^{\prime ,}$, $d^{\prime ,}$, etc. mean respectively the entire difference and differential of xy, relatively to x only; $\Delta^{,\prime}$, $d^{,\prime}$, relatively to y only; for instance, $d(xy)=xdy+ydx=d^{,\prime}(xy)+d^{\prime ,}(xy)=(d^{,\prime}+d^{\prime ,})xy$: The advantage resulting from this change of notation,

cannot be here distinctly pointed out, but it would be difficult, I apprehend, to introduce a similar and corresponding change in the fluxionary notation: and this is an additional reason, why I have made a deviation, for which the English reader may feel a propensity to blame me."

604. *A. L. Crelle.*—In this restless period in the development of the calculus, there were many minds, and divided councils. We have seen that new notations were suggested by Crelle,[1] who in partial derivatives has a notation resembling Euler's of 1776; Crelle states (p. 72): "When $z=|xy|$, then we let $\frac{d}{x}z$ or $\frac{d}{x}|xy|$, or $\frac{d}{x}fxy$ stand for the first partial derivation according to x of the magnitude $|xy|$, while $\frac{d}{y}z$ or $\frac{d}{y}|xy|$ or $\frac{d}{y}fxy$ stands for the partial derivation according to y, of the magnitude $|xy|$." Similarly, $\frac{d^2}{x^2}z$ or $\frac{d^2}{x^2}fxy$ or $\frac{d^2}{x^2}|xy|$ indicates the second partial derivative with respect to x, while $\frac{d^2}{xy}z$ represents the partial derivative with respect to x, followed by that with respect to y. It is to be noted that with Crelle d does not signify "differential," but *abgeleitete Grösse*, which Lagrange called the "derived function" and Arbogast the "derivation." The partial is distinguished from the total derivative, by having the independent variable with respect to which the differentiation takes place written below the d. Thus dz means the total derivative, $\frac{d}{x}z$ means the partial. On page 82 he introduces also the D for the total derivative.

605. *P. Barlow.*—A quaint notation was given in England in 1814 by Peter Barlow.[2] If u is any function of x, y, z, etc., let δ, d, D, etc., be the characteristics of their differentiation relatively to each of the variables, then the partial differences are shown here in the second and third lines,

$$\begin{array}{cccc} x\,, & y\,, & z\,, & \text{etc.,} \\ \dfrac{\delta u}{dx}\,, & \dfrac{du}{dy}\,, & \dfrac{Du}{dz}\,, & \text{etc.,} \\ \dfrac{\delta\delta u}{dx^2}\,, & \dfrac{d\delta u}{dx\,dy}\,, & \dfrac{D\delta u}{dx\,dz}\,, & \text{etc.} \end{array}$$

[1] A. L. Crelle, *Rechnung mit veränderlichen Grössen*, Vol. I (Göttingen, 1813).

[2] P. Barlow, *Dictionary of Mathematics* (1814), art. "Partial Differences."

The association of x, y, z, etc., with δ, d, D, etc., respectively, makes unnecessarily heavy demands upon the memory of the reader.

606. *A. L. Cauchy.*—Cauchy used a variety of different notations for partial differentiation at different times of his long career. In 1823, in his lessons on the infinitesimal calculus,[1] he represents partial differentials by $d_x u$, $d_y u$, $d_z u$, and the partial derivatives by $\frac{d_x u}{dx}$, $\frac{d_y u}{dy}$, $\frac{d_z u}{dz}$, but remarks that the latter are usually written, for brevity, $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$. He writes also the more general partial derivative $\frac{d_x^l\, d_y^m\, d_z^n \ldots . u}{dx^l\, dy^m\, dz^n \ldots .}$, but adds that the letters x, y, z at the base of the d's are ordinarily omitted, and the notation $\frac{d^{l+m+n} \ldots . u}{dx^l\, dy^m\, dz^n \ldots .}$ is used. The same notation, and similar remarks on partial derivatives and partial differentials, are found in his lessons on the differential calculus[2] of 1829.

In England, John Hind[3] gives the general form of partial derivatives which we last quoted for Cauchy, and then remarks: "Another kind of notation attended with some conveniences has of late been partially adopted. In this, the differential coefficients are no longer expressed in a fractional form, but are denoted by the letter d with the principal variables suffixed: thus, $\frac{du}{dx}$ and $\frac{du}{dy}$ are equivalent to $d_x u$ and $d_y u$," and so that the foregoing partial derivative of the order $l+m+n+$etc. is written $d_x^l d_y^m d_z^n$ etc. u. This notation which Hind describes is that of Cauchy for partial differentials.

De Morgan, when considering partial processes, lets $\frac{du}{dx}$ stand for a partial derivative, and $\frac{d \cdot u}{dx}$ for a total derivative, the two being "totally distinct."[4]

[1] Cauchy, *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal* (Paris, 1823); *Œuvres* (2d ser.), Vol. IV, p. 50, 79.

[2] Cauchy, *Leçons sur le calcul différentiel* (Paris, 1829); *Œuvres* (2d ser.), Vol. IV, p. 513, 527.

[3] John Hind, *Principles of the Differential Calculus* (2d ed.; Cambridge, 1831), p. 372.

[4] Augustus De Morgan, *Differential and Integral Calculus* (London, 1842), p. 88–91.

607. *M. Ohm.*—The much-praised and much-criticized Martin Ohm,[1] in Germany, advanced notations as follows:

If f is a function of $x, y, z, \ldots\ldots$, then the partial derivatives with respect to x are indicated by $\partial f_x, \partial^2 f_x, \partial^3 f_x, \ldots\ldots, \partial^a f_x$. Similarly for partial derivatives with respect to y or z, etc. More generally, $\partial^{a,b,c} f_{x,y,z}$ means the ath partial derivative with respect to x, followed by the bth partial derivative with respect to y, and then by the cth partial derivative with respect to z. Ohm writes the form $\partial^{o,a,b} f_{x,y,z}$ and also $\partial^{a,b} f_{y,z}$; the latter notation is shorter but does not display to the eye which of the variables is constant. Ohm's is perhaps the fullest notation for partial differentiation developed up to that time. But Ohm was not content to consider simply functions which are expressed directly and explicitly in terms of their variables; he studied also functions $f_{x,y,z}, \ldots\ldots$, in which some of the variables $x, y, z, \ldots\ldots$, appear implicitly, as well as explicitly. For instance (p. 129), let $f=3x^2+4xy-5y^2z^2+4xyz$, where $y=a+bx$ and $z=px^3$. He introduces the notation $f_x=3x^2+4xy-5y^2z^2+4xyz$, where f_x is considered as the explicit function of x, and y and z are to be taken constant or independent of x; he represents by $f_{(x)}$ the foregoing expression when account is taken of $y=a+bx$ and $z=px^3$. Hence $\partial f_x=6x+4y+4yz$, while $\partial f_{(x)}=4a+(8b+6)x-30a^2p^2x^5-70abp^2x^6-40b^2p^2x^7+16apx^3+20bpx^4$. The two partial derivatives are wholly different; the former is an incomplete partial derivative, the latter is a complete partial derivative. He states (p. 276) that in 1825 he brought out in Berlin his *Lehre vom Grössten und Kleinsten*, and introduced there a special notation for the complete and incomplete partial differentials which correspond to the complete and incomplete partial derivatives $f_{(x)}$ and f_x. In 1825 he marked the incomplete partial differentials $\frac{df}{dv} \cdot dv$; the complete partial differentials $\overset{df}{\underset{dv}{=}} \cdot dv$. Similarly for higher orders. Ohm states in 1829 (p. 276) that he has found his notation $\overset{df}{\underset{dx}{=}} \cdot dv$ adopted by other writers.

In the 1829 publication Ohm considers a great many cases of this type and gives a diffuse presentation of partial differentiation covering over one hundred and sixty pages. He compares his notations with those of Euler and Fontaine. $\frac{df}{dv}$ meant with Euler a complete

[1] Martin Ohm, *System der Mathematik*, Vol. III (1829), p. 118, 119.

derivative; with Fontaine, a partial derivative. As previously seen, Fontaine once marked a complete derivative $\frac{1}{dx}\cdot d\mu$, but generally he dealt with complete differentials rather than complete derivatives.

If f is a function of x and y, then the three notations are exhibited by Ohm as follows (p. 183):

$$df = df_x + df_y \qquad \text{(according to M. Ohm)}\ ,$$

$$df = \left(\frac{df}{dx}\right)\cdot dx + \left(\frac{df}{dy}\right)\cdot dy \qquad \text{(according to L. Euler)}\ ,$$

$$df = \frac{df}{dx}\cdot dx + \frac{df}{dy}\cdot dy \qquad \text{(according to A. Fontaine)}\ ,$$

where df means the complete differential. Ohm warns the reader (p. 182) that in Fontaine's notation for differentials the dx's and dy's cannot be canceled. We see in Ohm's intellectual *ménage* a large progeny of symbols, but it was a group with no one to bid them welcome.

608. *W. R. Hamilton.*—About the same time that Ohm was devising suitable symbols in Germany, William Rowan Hamilton in Great Britain felt the need of a suitable symbolism. At first he used parentheses[1] as Euler had done; they are found in an article which was written in 1824 and printed in 1828. In the table of contents of this article, probably prepared when it was printed, the partial derivatives appear in the garb $\frac{\delta i}{\delta x}$. This delta notation is used by Hamilton[2] in an article printed in 1830, and another printed in 1834,[3] where the expressions are called "partial differential coefficients." On account of the close resemblance between δ and ∂, this notation is almost identical with that proposed by Legendre forty years earlier.

609. *W. Bolyaĭ.*—Quaint notations are found in Wolfgang Bolyai's[4] *Tentamen* (1832). He designates by Φx a function of x, by ${}^{n}_{\emptyset}\!\!\mid \Phi x$ the nth differential, by ${}^{n}_{\emptyset}\!\!\mid \Phi x, y$ or $\mid^{n}_{\emptyset x} \Phi x, y$ the nth partial differential with respect to x.

[1] W. R. Hamilton in *Transactions of the Royal Irish Academy*, Vol. XV (Dublin, 1828), p. 152.

[2] *Op. cit.*, Vol. XVI (1830), p. 2.

[3] *Philosophical Transactions of the Royal Society* (London, 1834), p. 249.

[4] Wolfgangi Bolyai de Bolya, *Tentamen* (2d ed., 1897), p. 207, 640.

610. *Cauchy and Moigno.*—Cauchy slightly modified his former symbols. In 1840 L'Abbé Moigno[1] published a book on the calculus "d'après les méthodes et les ouvrages publiés ou inédits de M. A.-L. Cauchy." For higher partial differentials and derivatives he gives the notations $d_x^2 d_y d_z u$, $d^4_{x^2yz}$, $\frac{d^4 u}{dx^2\,dy\,dz}$. That these notations do not always distinguish the partial derivative from the total is evident from what is given on page 144 as partial derivatives: $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{d^2z}{dx^2}$, etc. That Moigno was not satisfied with his notations is evident from our previous quotations from his Introduction.

In the same year (1840) Cauchy himself brought out in Paris his *Exercices d'analyse et de physique mathématique*, where (p. 5) he lets D_x, D_y, D_z, D_t stand for the partial derivatives of a function of the independent variables x, y, z, t; D_t^2 stands for the second partial derivative.

More fully developed is his notation[2] of 1844. Taking s as a function of many variables, $x, y, z, \ldots, u, v, w$, Cauchy designates:

Partial increments	$\Delta_1 s$, $\Delta_{11} s$, $\Delta_{111} s$, ...
Différentielles partielles	$d_1 s$, $d_{11} s$, $d_{111} s$, ...
If of the second order	$d_1 d_{11} s$, $d_1 d_{111} s$ ....
When the variables are specified	$\Delta_x s$, $\Delta_y s$, $\Delta_z s$, ...
Différentielles partielles	$d_x s$, $d_y s$, $d_z s$, ...
When of the second order	$d_x d_y s$, $d_x d_z s$, ...
Dérivées partielles	$D_x D_y s$, $D_x D_z s$, ...
Etc.	

The $D_x s$ as a partial derivative is distinguished from a total derivative by the x, provided that s is a function of more than one variable; otherwise the $D_x s$ and Ds mean the same. One objection to this symbolism is that the variables considered as constant during the differentiation are not exhibited to the eye.

611. *C. G. J. Jacobi.*—We come now to the researches of C. G. J. Jacobi who is usually credited with the invention of the notation of partial derivatives which has become popular in recent years. Had he instituted a careful historical survey of the notations which had been proposed by mathematicians before him, he would have found much of

[1] L'Abbé Moigno, *Leçons de calcul différentiel et de calcul intégral*, Tome I (Paris, 1840), p. 118–21.

[2] A. L. Cauchy, *Exercices d'analyse et de physique mathématique* (Paris, 1844), p. 12–17.

value—notations of Euler, Fontaine, Karsten, Legendre, Lagrange of 1797 and of 1801, Da Cunha, Arbogast, Crelle, Barlow, Ohm, Hamilton, and Cauchy. But Jacobi made no such survey. He happened to know the notations of Euler and of Lagrange of 1797. Not satisfied with these he devised one of his own. As is so common in the history of science, his invention happened to have been made by others before him. Nevertheless, the mathematical public, uncritical on matters of priority, is crediting Jacobi with devices first proposed by Legendre and Hamilton.

It was in 1841 that Jacobi published his paper[1] "De determinantibus functionalibus," in which he introduces d and ∂ for total and partial derivatives (*differentialia partialia*), respectively. He says: "To distinguish the partial derivatives from the ordinary, where all variable quantities are regarded as functions of a single one, it has been the custom of Euler and others to enclose the partial derivatives within parentheses. As an accumulation of parentheses for reading and writing is rather onerous, I have preferred to designate ordinary differentials by the character d and the partial differential by ∂. Adopting this convention, error is excluded. If we have a function f of x and y, I shall write accordingly,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\ .$$

". . . . This distinction may be used also in the designation of integrations, so that the expressions

$$\int f(x,y)dx\ , \qquad \int f(x,y)\partial x$$

have different meaning; in the former y and therefore also $f(x, y)$ are considered functions of x, in the latter one carries out the integration according to x only and one regards y during the integration as a constant."

Jacobi proceeds to the consideration of a function of many variables. "Let f be a function of $x, x_1, \ldots\ldots, x_n$. Assume any n functions $w_1, w_2, \ldots\ldots, w_n$ of these variables and think of f as a function of the variables $x, w_1, w_2, \ldots\ldots, w_n$. Then, if $x_1, x_2, \ldots\ldots, x_n$ remain constant, the $w_1, w_2, \ldots\ldots, w_n$ are no longer constants when x varies, and also, when $w_1, w_2, \ldots\ldots, w_n$ remain constant, $x_1, x_2, \ldots\ldots, x_n$ do not remain constant. The expression $\frac{\partial f}{\partial x}$ will designate entirely different

[1] *Journal für die reine und angewandte Mathematik*, Vol. XXII, p. 319; also Vol. XXIII, p. 4. See *Ostwald's Klassiker*, No. 78.

values according as these or other magnitudes are constant during the differentiation." To simplify matters, Jacobi stipulates: "If I say that f is a function of the variables $x, x_1, \ldots, x_n$, I wish it to be understood that, if this function is differentiated partially, the differentiation shall be so carried out that always only one of these varies, while all the others remain constant."

Jacobi proceeds: "If the formulas are to be free from every ambiguity, then also there should be indicated by the notation, not only according to which variable the differentiation takes place, but also the entire system of independent variables whose function is to be differentiated partially, in order that by the notation itself one may recognize also the magnitudes which remain constant during the differentiation. This is all the more necessary, as it is not possible to avoid the occurrence in the same computation and even in one and the same formula, of partial derivatives which relate to different systems of independent variables, as, for example, in the expression given above, $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}$, in which f is taken as a function of x and u, u on the other hand is a function of x and y. The $\frac{\partial f}{\partial x}$ [derived from $f(x, y)$] changed into this expression when u was introduced in the place of y as an independent variable. If however we write down, besides the dependent variable, also the independent variables occurring in the partial differentiations, then the above expression can be represented by the following formula which is free of every ambiguity:

$$\frac{\partial f(x, y)}{\partial x}=\frac{\partial f(x, u)}{\partial x}+\frac{\partial f(x, u)}{\partial u}\cdot\frac{\partial u(x, y)}{\partial x}\text{.''}$$

By way of criticism of this notation, P. Stäckel[1] adds in his notes on Jacobi's paper: "This notation is ambiguous too, for the symbol is used in two different meanings, in as much as $f(x, y)$ is another function of x and y than is $f(x, u)$ of x and u." Jacobi proceeds to remark that one can conceive of complicated relations in which it would be exceedingly onerous completely to define partial derivatives by the notation alone; a formula written in only one line might have to be expanded so as to fill a page. To avoid such differences it is best in complicated cases to dispense with the designation in the formula of all the independent variables. To distinguish two systems of partial derivatives which belong to different systems of variables, one system

[1] *Ostwald's Klassiker*, No. 78, p. 65.

of derivatives may be inclosed in parentheses, as was the practice with Euler.

The notation $\frac{\partial u}{\partial x}$ advocated by Jacobi did not meet with immediate adoption. It took half a century for it to secure a generally recognized place in mathematical writing. In 1844 Jacobi used it in Volume XXIX of *Crelle's Journal.* When in 1857 Arthur Cayley[1] abstracted Jacobi's paper, Cayley paid no heed to the new notation and wrote all derivatives in the form $\frac{du}{dx}$.

G. A. Osborne refers in the Preface of his *Differential and Integral Calculus* (Boston, 1895) to this notation as having "recently come into such general use."

612. *O. Hesse.*—A few years after Jacobi had published the article containing the $\frac{\partial z}{\partial x}$ notation, Otto Hesse advanced another notation which has been found useful in analytical geometry. In his paper on points of inflection of cubic curves[2] he designates $\frac{\partial^2 u}{\partial x_\chi \partial x_\lambda}$ of the function u for brevity by $u_{\chi,\lambda}$. In accordance with this plan Salmon[3] writes the first partial derivatives of U with respect to x, y, and z, respectively, by U_1, U_2, U_3 and the second partial derivatives by U_{11}, U_{22}, U_{33}, U_{23}, U_{31}, U_{12}; or, as it is expressed in Fiedler's edition, U_i is the first partial derivative of U with respect to x_i.

613. *B. Peirce.*—Arbogast's and Cauchy's symbolism was in part adopted in the United States by Benjamin Peirce,[4] of Harvard, who introduced it thus: "Denoting by $D\cdot_x\cdot$, $D\cdot_u\cdot$, the differential coefficients taken on the supposition that x, u are respectively the independent variables, we have, at once, $D\cdot{}_x y=\frac{dy}{dx}=\frac{D\cdot_u y}{D\cdot_u x}$." Accordingly, D signifies the total derivative, $D\cdot_x y$ the partial derivative, with respect to x.

From an anonymous article in the *Mathematical Monthly* (probably from the pen of its editor, J. D. Runkle)[5] we cull the following comments:

[1] A. Cayley, "Report on Recent Progress in Theoretical Dynamics," *British Association Report,* 1857 (London, 1858), p. 24.

[2] Otto Hesse in *Crelle's Journal* (1843), Vol. XXVIII, p. 99; *Gesammelte Werke* (München, 1897), p. 124.

[3] George Salmon, *Higher Plane Curves* (3d ed.; Dublin, 1879), p. 49, 50.

[4] B. Peirce, *Curves, Functions, and Forces,* Vol. I (2d ed.; Boston, 1852), p. 225.

[5] *Mathematical Monthly* (ed. J. D. Runkle), Vol. I (Cambridge, 1859), p. 328.

"The student will observe, that $\frac{du}{dx}$ denotes the derivative of the quantity u; but the symbol, as separated from the quantity, and simply denoting the operation, is $\frac{d}{dx}$. Thus, $\frac{d}{dx}f(x)$ tells us to find the derivative of $f(x)$. The inconvenience of the use of the symbol $\frac{d}{dx}$, in this and like cases has led to the adoption of D in its place. If we wish to indicate at the same time the particular variable, as x, in reference to which the derivative is to be taken, then the symbol D_x is used. . . . . In the case of the general function, as $f(x)$, the notation $f'(x)$, $f''(x)$, $f'''(x)$, etc. to denote the successive derivatives, was used by Lagrange, and is most convenient. So far as we know, Prof. Peirce is the only author in this country who has used D; and we have made these remarks for the benefit of those students who meet with this notation only in the *Monthly*. . . . . Dx^n is better than $\frac{d}{dx}x^n$."

614. *G. W. Strauch.*—A special symbolism to distinguish between two partial derivatives with respect to x, one recognizing only explicit forms of x, the other observing also the implicit forms, was given by the Swiss, G. W. Strauch.[1] He assumes $U=\varphi(x, y, z)$ and $z=\psi(x, y)$. Following in part Cauchy's notation, Strauch uses the symbol $\frac{d_xU}{dx}$ to represent the "incomplete" partial differential coefficient with respect to x which is obtained by differentiating U only with respect to the explicit x; he uses the sybmol $\frac{d_xU}{\overline{\overline{dx}}}$ to represent the "complete" partial differential coefficient with respect to x which is obtained by differentiating U with respect to the implicit as well as the explicit x. He has the following relation between the partial derivatives,

$$\frac{d_xU}{\overline{\overline{dx}}}=\frac{d_xU}{dx}+\frac{d_zU}{dz}\cdot\frac{d_xz}{dx}\,.$$

It will be remembered that the double horizontal line used here had been employed for the same purpose in 1825 by Ohm (§ 607).

615. *J. M. C. Duhamel.*—The D_x, D_y, etc., suggested by Cauchy for partial differentiation was used in many quarters. For example,

[1] G. W. Strauch, *Praktische Anwendungen für die Integration der totalen und partialen Differentialgleichungen* (Braunschweig, 1865), p. 7–13.

Ernest Pfannenstiel,[1] in 1882, wrote $D_x^2 z$, $D_{x,y}^2 z$, etc., to represent partial derivatives in a paper on partial-differential equations.

Duhamel[2] in ordinary cases used no distinctive device for partial differentiation, but when many variables and derivatives of any orders were involved, he designated "par $F_{x,y,x\ldots\ldots}^{m+n+p\ldots\ldots}(x, y)$ ou par $D_{x,y,x\ldots\ldots}^{m+n+p\ldots\ldots}u$ le résultat de m dérivations partielles effectuées par rapport à x sur la fonction $u=F(x, y)$, suivies de n dérivations partielles du résultat par rapport à y, lesquelles seront elles-mêmes suivies de p dérivations par rapport à x; et ainsi de suite." For partial differentials he wrote similarly, $d_{x,y,x\ldots\ldots}^{m+n+p\ldots\ldots}u$. It will be observed that, strictly speaking, there is nothing in this notation to distinguish sharply between partial and total operations; cases are conceivable in which each of the $m+n+p$ .... differentiations might be total.

616. *G. S. Carr*[3] proposed "experimentally" a notation corresponding to that for total derivation; he would write $\frac{\partial^5 u}{\partial x^2 \partial y^3}$ in the contracted form $u_{2x\,3y}$.

617. *Ch. Meray.*—An obvious extension of the notation used by Duhamel was employed by Ch. Méray.[4] Representing by $p, q, \ldots\ldots$, the orders of the partial derivatives with respect to the independent variables $x, y, \ldots\ldots$, respectively, and by the sum $p+q+\ldots\ldots$ the total order of the derivatives, he suggests two forms, either

$$f_{x,y,\ldots\ldots}^{(p,q,\ldots\ldots)}(x, y, \ldots\ldots) \qquad \text{or} \qquad D_{x,y,\ldots\ldots}^{(p,q,\ldots\ldots)}f(x, y, \ldots\ldots)\,.$$

Applying this notation to partial differentials, Méray writes,

$$d_{x,y,\ldots\ldots}^{(p,q,\ldots\ldots)}f=f_{x,y,\ldots\ldots}^{(p,q,\ldots\ldots)}(x, y, \ldots\ldots)\,dx^p\,dy^q\ldots\ldots$$

Méray's notation is neither elegant nor typographically desirable. It has failed of general adoption. In the *Encyclopédie des sciences mathématiques*[5] J. Molk prefers an extension of the Legendre-Jacobi notation; thus $\frac{\partial^k u}{\partial x^p\,\partial y^q\ldots\ldots}$, where $k=p+q+\ldots\ldots$

[1] *Societät der Wissenschaften zu Upsala* (Sept. 2, 1882).

[2] J. M. C. Duhamel, *Eléments de calcul Infinitésimal*, Tome I (3d ed.; Paris, 1874), p. 267.

[3] G. S. Carr, *Synopsis .... of the Pure Mathematics* (London, 1886), p. 266, 267.

[4] Ch. Méray, *Leçons nouvelles sur l'analyse infinitésimale*, Vol. I (Paris, 1894), p. 123.

[5] Tome II, Vol. I, p. 296.

618. *T. Muir.*—Thomas Muir stated[1] in 1901 that he found it very useful in lectures given in 1869 to write $\Phi\overline{x, y, z}$ in place of $\Phi(x, y, z)$ and to indicate the number of times the function had to be differentiated with respect to any one of the variables by writing that number above the vinculum; thus, $\Phi\overset{1\ 3\ 2}{\overline{x,y,z}}$ meant the result of differentiating once with respect to x, thrice with respect to y, and twice with respect to z. Applying this to Jacobi's example, in which $z=\Phi\overline{x, y}$, we should have satisfactorily $\frac{\partial z}{\partial x}=\Phi\overset{1\quad}{\overline{x, y}}$; but when there is given $z=\Phi\overline{x, u}$ and $u=\Psi\overline{x, y}$, then arises the serious situation that we are not certain of the meaning of $\frac{\partial z}{\partial x}$, as it would stand for $\Phi\overset{1\quad}{\overline{x, u}}$ or for $\Phi\overset{1\quad}{\overline{x, u}}+\Phi\overset{\quad 1}{\overline{x, u}}\cdot\Psi\overset{1\quad}{\overline{x, y}}$, according as u or y was to be considered constant. In such cases he declares the notation $\frac{\partial z}{\partial x}$ inadequate.

Thomas Muir explained his notation also in *Nature*, Volume LXVI (1902), page 271, but was criticized by John Perry who, taking a unit quantity of mere fluid, points out that v, p, t, E, Φ, are all known if any two (except in certain cases) are known, so that any one may be expressed as a function of any other two. Perry prefers the notation $\left(\frac{dE}{dv}\right)_p$, which is the Eulerian symbol to which p is affixed; according to Muir's suggestion, Perry says, one must let $E=f\overline{v, p}$ and write $f\overset{1\quad}{\overline{v, p}}$. To this remark, Muir replies (p. 520) that he would write $E\overset{1\quad}{\overline{v, p}}$, or, if a vinculum seems "curious," he would write $E(\overset{1}{v}, p)$. To these proposals A. B. Basset remarks (p. 577): "Dr. Muir's symbols may be very suitable for manuscripts or the blackboard, but the expense of printing them would be prohibitive. No book in which such symbols were used to any extent could possibly pay. On the other hand, the symbol $(dE/dv)_p$ can always be introduced into a paragraph of letter-press without using a justification or a vinculum; and this very much lessens the expense of printing."

In the case of $u=f(x, y)$ and $y=\phi(x, z)$, some authors, W. F. Os-

[1] Thomas Muir in *Proceedings of the Royal Society of Edinburgh*, Vol. XXIV (1902–3), p. 162.

good[1] for instance, write $\left.\frac{\partial u}{\partial x}\right]_y$ to indicate that y is held fast, and $\left.\frac{\partial u}{\partial x}\right]_z$ to indicate that z is held fast.

619. *P. Mansion.*—Muir and Perry were not alone in their rejection of the notation used by Jacobi in 1841. Paul Mansion,[2] for instance, in 1873 prefers to use the straight letter d for partial derivatives; thus, $\frac{dz}{dx_1}, \frac{dz}{dx_2}$, where z is an explicit or implicit function of the independent variables $x_1, x_2, \ldots.$ He uses also $\frac{\delta\varphi}{\delta x_1}, \frac{\delta\varphi}{\delta x_2}$ to designate "the derivatives of an *explicit* function $\varphi(x_1, x_2, \ldots.)$ of $x_1, x_2, \ldots.$ with respect to the letter X_1, to the letter X_2, etc., without considering whether $x_1, x_2, \ldots.$ are independent of each other or not."

3. Symbols for Integrals

620. *G. W. Leibniz.*—In a manuscript[3] dated October 29, 1675, as previously noted (§§ 544, 570), Leibniz introduces the symbol $\int l$ for *omn. l*, that is, for the sum of the l's. As a facsimile reproduction was published by C. I. Gerhardt in 1899, we have definite information as to the exact form of the $\int$. It was the long form of the letter s, frequently used at the time of Leibniz (see Fig. 124). Not until eleven years later did Leibniz use the symbol in print. In 1686 Leibniz used it eight times in an article in the *Acta eruditorum* (p. 297, 299), but in that paper it did not quite take the form found in his manuscript and in later printing; the lower part was amputated and the sign appeared thus, ſ. It was simply the small letter s, as printed at that time. It resembled the modern type form for "f." He writes the cycloidal formula thus: $y = \sqrt{2x - xx} + ſ\, dx : \sqrt{2x - xx}$. This form of the symbol appears in the *Acta eruditorum* for 1701 (p. 280), where Louis Carré, of Paris, writes $a\, ſ\, dx = ax$, and again in 1704 (p. 313), in a reprint of John Craig's article from the *Philosophical Transactions of London*, No. 284, viz., $ſ : z\, dy$. We have noticed this form also in works of Manfredi[4] at Bologna in 1707 and of Wolf[5] at Halle in 1713.

[1] W. F. Osgood, *Differential and Integral Calculus* (New York, 1907), p. 306.

[2] *Théorie des équations aux dérivées partielles du premier ordre*, par M. Paul Mansion, professeur à l'université de Gand (1873), Tome XXV.

[3] C. I. Gerhardt, *Entdeckung der höheren Analysis* (Halle, 1855), p. 125.

[4] *De Constructione aequationum differentialium primi gradus.* Authore Gabriele Manfredio (Bononiae, 1707), p. 127.

[5] *Elementa matheseos universae,* tomus I . . . . Autore Christiano Wolfo (Halae, 1713), p. 474.

In *Leibnizens Mathematische Schriften* (ed. C. I. Gerhardt), Volume V (1858), page 231, the notation of the article of 1686 is not reproduced with precision; the sign of integration assumes in the reprint the regular form $\int$, namely, a slender, elongated form of the letter. John Bernoulli,[1] who used the term "integral" (first employed by Jacob Bernoulli, see § 539), proposed in his correspondence with

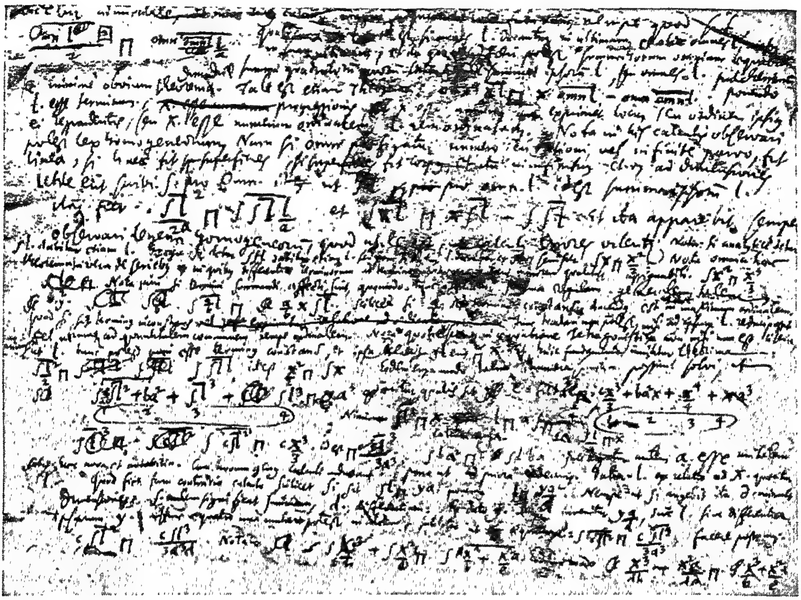


FIG. 124.—Facsimile of manuscript of Leibniz, dated Oct. 29, 1675, in which his sign of integration first appears. (Taken from C. I. Gerhardt's *Briefwechsel von G. W. Leibniz mit Mathematikern* [1899].)

Leibniz the letter I as the symbol of integration but finally adopted $\int$ in deference to Leibniz. We cannot agree with Klügel,[2] that I would have been more appropriate (*schicklicher*); this shorter symbol possessed less adaptability to the nineteenth-century need of indicating symbolically the limits of integration. In articles published in the *Acta eruditorum* of 1694 and 1695, Leibniz places a comma after the

[1] Johann Bernoulli to Leibniz, April 7, 1696, *Leibnizens Mathematische Schriften* (ed. C. I. Gerhardt), Vol. III (Halle, 1855), p. 262, 273.

[2] G. S. Klügel, *Mathematisches Wörterbuch*, 1. Abth., 2. Theil (Leipzig, 1805), art. "Integral," p. 746.

$\int,$ thus, $\int, xx\,dx$, but Johann Bernoulli omits the comma in the volume for 1698 (p. 53). Some eighteenth- and early nineteenth-century writers placed a dot after the sign, thus, $\int.$, or the colon, as in $\int:$; E. Waring[1] wrote $\int.\frac{\dot{x}}{x}(\log.\, x)$. In 1778 Euler[2] wrote $\int p\,dx\int q\,dx\int r\,dx$ ...., which did not indicate a product like $\int p\,dx\cdot\int q\,dx\cdot\int r\,dx$ ...., but was to be interpreted so that $\int$ applies to all that follows—a relation which some authors (Gherli, for instance, as we shall see) carefully indicate by the use of vinculums.

As regards the differential in the expression to be integrated, the usual practice has been to write it down, thus, $\int y\,dx$, but this practice has not been universal. Leibniz' conception of an integral as a summation would seem to require the use of the differential. In his printed paper of 1686 he adopts the form $\int p\,dy$, although in his manuscript of October 29, 1675, he had written $\int x^2 = \frac{x^3}{3}$. If integration is conceived as the converse of differentiation, then no serious objection can be raised to the omission of the differential. And so we find a few authors who prefer to omit the dx, as, for instance, Benjamin Peirce,[3] who writes $(n+1)\int.\, ax^n = ax^{n+1}$. On October 11–16, 1858, William R. Hamilton wrote to P. G. Tait: "And perhaps you may have adopted, even publicly—as Airy has done, using the (to me) uncouth notation $\int_\theta(\quad)$ for $\int(\quad)d\theta$—the system which rejects differentials."[4]

4. Early Use of Leibnizian Notation in Great Britain

621. Some years before Newton permitted his notation for fluxions and fluents to see the light of day in printed form, his friend, John Craig, used the notation of Leibniz, dp, dx, dy, in a book, the *Methodus figurarum*, published in 1685 in London. Craig employed the Leib-

[1] Edward Waring, *Meditationes analyticae* (1786), p. 23.

[2] L. Euler, *Institutiones Calculi Integralis*, Vol. IV, Suppl. IX, art. 37.

[3] B. Peirce, *Curves, Functions, and Forces*, Vol. II (Boston, 1846), p. 69.

[4] C. G. Knott, *Life and Scientific Work of Peter Guthrie Tait* (Cambridge, 1911), p. 121. For this reference and also some other data in this article I am indebted to Professor J. H. M. Wedderburn.

nizian notation again in 1693 in another booklet, the *Tractatus mathematicus*, as well as in articles printed in the *Philosophical Transactions of London* for the years 1701, 1703, 1704, 1708; his article of 1703 contains the sign of integration $\int$. In the *Transactions* for 1704–5, an article by Johann Bernoulli makes extensive use of the Leibnizian signs. In 1716 the English physician and philosophical writer, George Cheyne, brought out in London the *Philosophical Principles of Religion*, Part II, which contains a chapter from the pen of Craig on a discussion of zero and infinity, dated September 23, 1713. In this chapter Craig uses Leibniz' symbols for differentiation and integration. But in 1718 Craig made a complete change. In that year he issued a book, *De calculo fluentium*, in which he switches over to the exclusive use of the Newtonian notation. Evidently this change is a result of the controversy then raging between the supporters of Newton and Leibniz.

In Volume XXIII of the *Philosophical Transactions of London*, for the years 1702 and 1703, De Moivre uses dots for fluxions, but in integration he uses the Leibnizian sign $\int$. Thus, on page 1125, De Moivre writes "adeoq; $\dot{q}=\frac{3}{2}dv^2\dot{v}-\frac{3}{2}dv^2\dot{y}$, igitur $q=\frac{1}{2}dv^3-\int\frac{3}{2}dv^2\dot{y}$. Ergo ad hoc perventum est ut fluentum quantitatem inveniamus cujus fluxio est $\frac{3}{2}dv^2\dot{y}$." Evidently his partisanship favoring Newton did not at this time prevent his resorting to the convenience of using the Leibnizian symbol of integration.

Even John Keill, who fought so prominently and unskilfully on the side of Newton, employed a similar mixed notation. In a paper dated November 24, 1713, and printed in the *Philosophical Transactions* for 1714–16, he adopts the symbolism $\int\phi\dot{x}$. A mixture of continental and British symbols is found in the writings of Waring and much later in Olinthus Gregory's *Treatise of Mechanics* (3d ed.; London, 1815) where on page 158 there is given $\int dy\sqrt{(\dot{x}^2+\dot{y}^2)}$. Similarly, Playfair[1] wrote $\int\dot{x}\sin v$. John Brinkley[2] used the $\int$ with a horizontal bar added, as in $f\dot{p}s$. In times still more recent the Newtonian dot has been used to advantage, by the side of Leibnizian symbols, in the treatment of mechanics and the algebras of vectors.

Returning to the eighteenth century, we find Benjamin Robins[3]

[1] *The Works of John Playfair, Esq.*, Vol. III (Edinburgh, 1822), p. 16.

[2] *Transactions of the Royal Irish Academy*, Vol. XII (1815), p. 85; see also Vol. XIII (1818), p. 31.

[3] Benjamin Robins, *Remarks on Mr. Euler's Treatise on Motion, Dr. Smith's Compleat System of Opticks, and Dr. Jurin's Essay upon Distinct and Indistinct Vision* (London, 1739).

writing the Leibnizian signs in a review of a publication of L. Euler. Joseph Fenn, an Irish writer who had studied on the Continent, and was at one time professor of philosophy in the University of Nantes, issued at Dublin, soon after 1768, a *History of Mathematics*. In it Fenn had occasion to use the calculus, and he employs the Leibnizian notation. In a "Plan of the Instructions" given in the Drawing School established by the Dublin Society (p. lxxxix of the foregoing volume) he discusses the tides and uses the calculus and the notation of Leibniz. He uses them again in the second volume of the *Instructions* given in the Drawing School established by the Dublin Society (Dublin, 1772). He is friendly to Newton, uses the terms "fluxion" and "fluent," but never uses Newton's notation. He is perhaps the last eighteenth-century writer in Great Britain who used the symbolism of Leibniz in differentiation and integration. The Leibnizian notation was the earliest calculus notation in England which appeared in print, but in the latter part of the eighteenth century it vanished almost completely from British soil.

5. Symbols for Fluents; Later Notations in the Integral Calculus

622. *I. Newton.*—In his *Quadratura curvarum*,[1] of 1704, Newton gave the symbol $\overset{|}{x}$ as the integral of x, $\overset{||}{x}$ as the integral of $\overset{|}{x}$. In fact, in the succession, $\overset{||}{x}$, $\overset{|}{x}$, x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, each term was the fluxion (fluent) of the preceding (succeeding) term. Another notation for integral was the inclosure of the term in a rectangle,[2] as is explained in Newton's *De Analysi per equationes numero terminorum infinitas*, where $\boxed{\frac{aa}{64x}}$ stands for $\int \frac{aa \cdot dx}{64x}$.

That Newton's notation for integration was defective is readily seen. The $\overset{|}{x}$ was in danger of being mistaken for an abscissa in a series of abscissas x, x', x''; the rectangle was inconvenient in preparing a manuscript and well-nigh impossible for printing, when of frequent occurrence. As a consequence, Newton's signs of integration were never popular, not even in England. They were used by Brook Taylor, in his *Methodus incrementorum* (London, 1715); he writes (p. 2): "$\overset{||}{x}$ designat fluentem secondam ipsuis x" and (p. 38) "$\dot{p} = -r\dot{s}$, adeoque $p = -\boxed{r\dot{s}}$."

[1] I. Newton, *Quadratura curvarum* (London, 1704); *Opera*, Vol. I, p. 338.

[2] I. Newton, *Opera* (Horsley's ed.), Vol. I (1779), p. 272.

In Newton's *Principia* (1687), Book II, Lemma II, fluents are represented simply by capital letters and their fluxions by the corresponding small letters. Newton says: "If the moments of any quantities A, B, C, etc., increasing or decreasing, by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c, etc., the moment or mutation of the generated rectangle AB will be $aB+bA$." Here a velocity or fluxion is indicated by the same symbol as a moment. With Newton a "fluxion" was always a velocity, not an infinitely small quantity; a "moment" was usually, if not always, an infinitely small quantity. Evidently, this notation was intended only as provisional. Maclaurin does not use any regular sign of integration. He says[1] simply: "$\dot{y}z+\dot{z}y$, the fluent of which is yz." Nor have we been able to find any symbol of integration in Thomas Simpson's *Treatise of Fluxions* (London 1737, 1750), in Edmund Stone's *Integral Calculus*,[2] in William Hale's *Analysis fluxionum* (1804), in John Rowe's *Doctrine of Fluxions* (4th ed.; London, 1809), in S. Vince's *Principles of Fluxions* (Philadelphia, 1812). In John Clarke's edition of Humphry Ditton's text,[3] the letter F. is used for "fluent." Dominated by Wallis' concept of infinity, the authors state, "$F.\dfrac{\overset{n-1}{x}\;\dot{x}}{x^m}$, will be Finite, Infinite, or more than Infinite, according as n is $>$, $=$, or $<$ than m." We have seen (§ 582) that the letter F was used also for "fluxion."

623. *Ch. Reyneau and others.*—Perhaps no mathematical symbol has encountered so little competition with other symbols as has $\int$; the sign $\mathcal{S}$ can hardly be called a competitor, it being simply another form of the same letter. The $\mathcal{S}$ was used in France by Reyneau[4] in 1708 and by L'Abbé Sauri in 1774; in Italy by Frisi,[5] by Gherli,[6] who, in the case of multiple integrals, takes pains to indicate by vinculums

[1] C. Maclaurin, *Treatise of Fluxions*, Book II (1742), p. 600.

[2] We have examined the French translation by Rondet, under the title *Analyse des infiniment petits comprenant le Calcul Intégral*, par M. Stone (Paris, 1735).

[3] *An Institution of Fluxions* .... by Humphry Ditton (2d ed., John Clarke; London, 1726), p. 159, 160.

[4] Ch. Reyneau, *Usage de L'Analyse*, Tome II (Paris, 1708), p. 734.

[5] *Paulli Frisii Operum Tomus primus* (Milan, 1782), p. 303.

[6] O. Gherli, *Gli elementi teorico-pratici delle matematiche pure*, Tomo VI (Modena, 1775), p. 1. 334.

the scope of each integration, thus, $\overline{\overline{\overline{\mathbf{S}\cdot dx\int\cdot dx\int\cdot y\,dx}}}$. In Boscovich's[1] treatise of 1796 there is used part the time a small s, as in $s\,a\,x^m\,dx=\frac{ax^{m+1}}{m+1}$, then $\int y\,dx$, and finally $\mathbf{S}\frac{cy^2\,dx}{2r}$, this large letter being used, apparently, because there was plenty of space in front of the fraction. In the books which we have quoted the use of $\mathbf{S}$ rather than $\int$ was probably due to the greater plenty of the former type in the respective printing offices.

Not infrequently the two symbols were used in the same publication, the $\mathbf{S}$ to indicate some specialized integral. Thus, Fourier said in an article of 1811, first published in 1824: ". . . . And taking the integral from $x=0$ to $x=\pi$, one has, on representing these integrations by the sign $\mathbf{S}$, $\mathbf{S}\varphi \sin\cdot i\,x\,dx=$ . . . . ." Further on he writes "$a_0=\frac{1}{2}\mathbf{S}x\,dx$, ou $\frac{\pi^2}{4}$."[2] It will be seen that later Fourier suggested the notation for definite integrals now in general use. L. M. H. Navier,[3] in an article on fluid motion, remarks: "Le signe $\mathbf{S}$ désigne une intégration effectuée, dans toute l'entendu de la surface du fluide. ..." In the third edition of Lagrange's *Mécanique analytique*[4] we read: "Nous dénoterons ces intégrales totales, c'est-à-dire relatives à l'étendue de toute la masse, par la charactéristique majuscule $\mathbf{S}$, en conservant la charactéristique ordinaire $\int$ pour désigner les intégrales partielles ou indéfinies." Finally, we refer to C. Jordan, who, in his *Cours d'Analyse*, Volume I (Paris, 1893), page 37, uses $\mathbf{S}_E$ to mark "l'intégrale de la fonction f dans le domaine E," and says that it is generally designated thus.

[1] Ruggero Guiseppe Boscovich, *Elementi delle matematiche pure, Edizione terza Italiana* (Venezia, 1796), p. 477, 479, 484.

[2] *Mémoires de l'académie r. des sciences de L'Institut de France*, Tome IV, années 1819 et 1820 (Paris, 1824), p. 303, 309.

[3] *Ibid.*, année 1823, Tome VI (Paris, 1827), p. 412.

[4] K. L. Lagrange, *Œuvres*, Vol. XI, p. 85.

624. *A. L. Crelle*.—We have encountered only one writer on the European Continent who deliberately rejected Leibniz' sign $\int$ or $\mathfrak{S}$, namely, A. L. Crelle,[1] of Cassel, the founder of *Crelle's Journal*. In 1813 he rejected the symbol as foreign to the nature of the subject. He argued that since in differentiation (or the finding of dy, d^2y) d appears in the position of a multiplier, the symbol for the inverse operation, namely, integration, should be a d placed in the position of a divisor, thus, $\frac{1}{d}, \frac{1}{d^2}, \ldots$. Like Leibniz, Crelle was influenced in his selection by his conception of the operation called "integration." He looked upon it as the inverse of differentiation, and chose his symbol accordingly; Leibniz saw this inverse relation; nevertheless he looked upon integration primarily as a summation and accordingly chose the $\int$ as the first letter in *summa*. Crelle found no following whatever in his use of $\frac{1}{d}$.

625. *L. Euler*.—Limits of integration were at first indicated only in words. Euler was the first to use a symbol in his integral calculus,[2] of which the following is an illustration:

$$Q=\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} ab\; x^n = \frac{1}{2} \\ ad\; x \;= 1 \end{bmatrix} .$$

This notation, with the omission of the *ab* and *ad*, was used in 1819 or 1820 by F. Sarrus,[3] of Strasbourg, and H. G. Schmidten.[4]

626. *J. Fourier*.—Our modern notation for definite integrals constituted an important enrichment of the notation for integration. It was introduced by Joseph Fourier, who was an early representative of that galaxy of French mathematical physicists of the early part of the nineteenth century. In Fourier's epoch-making work of 1822 on *La Théorie analytique de la chaleur*[5] he says: "Nous désignons en génér-

[1] A. L. Crelle, *Darstellung der Rechnung mit veränderlichen Grössen*, Band I Göttingen, 1813), p. 88, 89.

[2] L. Euler, *Institutiones calculi integralis* (3d ed.), Vol. IV (1845), Suppl. V, p. 324. (First edition, 1768–70.)

[3] Gergonne, *Annales de mathématique*, Vol. XII (1819 and 1820), p. 36.

[4] *Op. cit.*, p. 211.

[5] *Œuvres de Fourier* (ed. G. Darboux), Tome I (Paris, 1888), p. 231; see also p. 216.

al par le signe $\int_a^b$ intégrale qui commence lorsque la variable équivalent à a, et qui est complète lorsque la variable équivaut à b; et nous écrivons l'équation (n) sous la forme suivante

$$\frac{\pi}{2}\varphi(x)=\tfrac{1}{2}\int_0^\pi \varphi(x)dx+\text{etc.''}$$

But Fourier had used this notation somewhat earlier in the *Mémoires* of the French Academy for 1819–20, in an article of which the early part of his book of 1822 is a reprint. The notation was adopted immediately by G. A. A. Plana,[1] of Turin, who writes $\int_0^1 a^u\,du=\frac{a-1}{Log.\ a}$; by A. Fresnel[2] in one of his memorable papers on the undulatory theory of light; and by Cauchy.[3] This instantaneous display to the eye of the limits of integration was declared by S. D. Poisson[4] to be a *notation très-commode*.[5]

It was F. Sarrus[6] who first used the signs $|F(x)|_a^x$ or $|_a^x F(x)$ to indicate the process of substituting the limits a and x in the general integral $F(x)$. This designation was used later by Moigno and Cauchy.

627. In Germany, Ohm advanced another notation for definite integrals, viz., $\int_{x\div a}\varphi.dx$ or $(\partial^{-1}\varphi_x)_{x\div a}$, where x is the upper limit and a the lower. He adhered to this notation in 1830 and 1846,

[1] J. D. Gergonne, *Annales de mathématiques*, Vol. XII (Nismes, 1819 and 1820), p. 148.

[2] *Mémoires de l'Académie r. des sciences de l'Institut de France*, Tome V, années 1821 et 1822 (Paris, 1826), p. 434.

[3] Cauchy, *Résumé des leçons données à l'école royale polytechnique sur la calcul infinitésimal* (Paris, 1823), or *Œuvres* (2d ser.), Vol. IV, p. 188.

[4] *Mémoires de l'académie des sciences de L'Institut de France*, Tome VI, année 1823 (Paris, 1827), p. 574.

[5] Referring to Fourier's notation, P. E. B. Jourdain said: "Like all advances in notation designed to aid, not logical subtlety, but rather the power possessed by mathematicians of dealing rapidly and perspicuously with a mass of complicated data, this improvement has its root in the conscious or unconscious striving after mental economy. This economical function naturally seems proportionally greater if we regard mathematics as a means, and not primarily as a subject of contemplation. It is from a mentally economical standpoint that we must consider Fourier's notation" (Philip E. B. Jourdain, "The Influence of Fourier's Theory of the Conduction of Heat on the Development of Pure Mathematics," *Scientia*, Vol. XXII [1917], p. 252).

[6] F. Sarrus in Gergonne's *Annales*, Vol. XIV (1823), p. 197

claiming for it greater convenience over that of Fourier in complicated expressions.[1]

Slight modification of the notation for definite integration was found desirable by workers in the theory of functions of a complex variable. For example, Forsyth[2] writes $_B\int$ where the integration is taken round the whole boundary B. Integration around a circle[3] is sometimes indicated by the sign $\oint$.

V. Volterra and G. Peano.—What C. Jordan[4] calls *l'intégrale par excès et par défaut* is represented by Vito Volterra[5] thus, $\int_{x_0}^{\overline{x_1}}$ and $\int_{\underline{x_0}}^{x_1}$.

G. Peano,[6] who uses $\mathcal{S}$ as the symbol for integration, designated by $\mathcal{S}(f, a \vdash\!\dashv b)$ the integral of f, extended over the interval from a to b, and by $\mathcal{S}^1(f, a \vdash\!\dashv b)$ the *intégrale par excès*, and by $\mathcal{S}_1(f, a \vdash\!\dashv b)$ the *intégrale par défaut.*

628. *E. H. Moore.*—In 1901, in treating improper definite integrals, E. H. Moore[7] adopted the notation for the (existent) $\begin{matrix}\text{narrow}\\ \text{broad}\end{matrix}$ Ξ—integral:

$$\int_{a(\Xi)}^{b} F(x)dx = \underset{D_I=0}{\overset{I|I(\Xi)}{L}} \int_a^b F_1(x)dx\ ,$$

$$\int_{a\{\Xi\}}^{b} F(x)dx = \underset{D_I=0}{\overset{I|I\{\Xi\}}{L}} \int_a^b F_1(x)dx\ ,$$

where Ξ is a point-set of points ξ, I is an interval-set, D_I is the length of I, $I(\Xi)$ is an interval-set which incloses Ξ narrowly (i.e.,

[1] Martin Ohm, *Lehre vom Grössten und Kleinsten* (Berlin, 1825); *Versuch eines vollkommen consequenten Systems der Mathematik*, Vol. IV (Berlin, 1830), p. 136–37; *Geist der Differential- und Integral-Rechnung* (Erlangen, 1846), p. 51 ff.

[2] A. R. Forsyth, *Theory of Functions of a Complex Variable* (Cambridge, 1893), p. 27.

[3] H. A. Kramers in *Zeitschrift für Physik*, Vol. XIII (1923), p. 346.

[4] C. Jordan, *Cours d'Analyse* (3d ed.), Vol. I (1909), p. 34, 35.

[5] V. Volterra, *Giornale di matematiche* (Battaglini), Vol. XIX (1881), p. 340.

[6] G. Peano, *Formulaire mathématique*, Vol. IV (Turin, 1903), p. 178.

[7] E. H. Moore, *Transactions of the American Mathematical Society*, Vol. II (1901), p. 304, 305.

every interval I contains or incloses at least one point ξ of Ξ), $I\{\Xi\}$ is an interval-set which incloses Ξ broadly (i.e., not necessarily narrowly), $F_1(x)$ has the value 0 or $F(x)$ according as the point x of the interval ab lies or does not lie on I.

629. *Residual calculus.*—In various papers Cauchy developed and applied a *calcul des résidus* which bears a certain analogy to the infinitesimal calculus. If $f(x)=\infty$ for $x=x_1$, and $\jmath(x)=(x-x_1)f(x)$, then $f(x)=\dfrac{f(x)}{x-x_1}$ when $x=x_1$, and $f(x_1)$ is called the residue of $f(x)$ with respect to x_1. Cauchy represents the operation of finding the residue by a special symbol. He says:[1] "Nous indiquerons cette extraction à l'aide de la lettre initiale $\mathcal{E}$, qui sera considérée comme une nouvelle caractéristique, et, pour exprimer le résidu intégral de $f(x)$, nous placerons la lettre $\mathcal{E}$ devant la fonction entourée de doubles parenthèses, ainsi qu'il suit: $\mathcal{E}((f(x)))$." Accordingly, $\mathcal{E}\dfrac{f(x)}{((F(x)))}$ stands for the sum of residues with respect to the roots of $F(x)=0$ only, while $\mathcal{E}\dfrac{((f(x)))}{F(x)}$ stands for the sum of the residues of $f(x)$ relative to $\dfrac{1}{f(x)}=0$ only. Furthermore,[2]

$$ {}_{x_0}^{X}\mathcal{E}_{y_0}^{Y}\,((f(z)))\ , $$

where $z=x+y\sqrt{-1}$ represents the residue of $f(z)$ taken between the limits $x=x_0$ and $x=X$, $y=y_0$ and $y=Y$. Laurent[3] employs the notation $\mathcal{E}_c f(z)$, where $f(c)=\infty$, or simply $\mathcal{E}f(z)$ when no ambiguity arises. B. Peirce[4] followed in the main Cauchy's notation. D. F. Gregory[5] changes the fundamental symbol to the inverted numeral 3, viz., ε, and distinguishes between the integral residues and the partial residues by suffixing the root to the partial symbol. Thus $\varepsilon_a f(x)$, $\varepsilon_b f(x)$ are partial, $\varepsilon f(x)$ is integral.

[1] A. L. Cauchy, *Exercices de mathématiques* (Paris, 1826); *Œuvres complètes* (2d ser.), Vol. VI, p. 26.

[2] Cauchy, *op. cit.*, Vol. VI, p. 256.

[3] H. Laurent, *Traité d'analyse* (Paris), Vol. III (1888), p. 243.

[4] Benjamin Peirce, *Curves, Functions, and Forces*, Vol. II (Boston, 1846), p. 43–59.

[5] D. F. Gregory, *Mathematical Writings* (Cambridge, 1865), p. 73–86; *Cambridge Mathematical Journal*, Vol. I (1839), p. 145.

6. Calculus Notations in the United States

630. The early influence was predominantly English. Before 1766 occasional studies of conic sections and fluxions were undertaken at Yale[1] under the leadership of President Clap. Jared Mansfield published at New Haven, Connecticut, about 1800, and again in 1802, a volume of *Essays, Mathematical and Physical,* which includes a part on "Fluxionary Analysis." In a footnote on page 207 he states that the fluxions of x, y, z are usually denoted "with a point over them, but we have here denoted these by a point somewhat removed to the right hand," thus, $x^{\cdot}$. Similar variations occur in Robert Woodhouse's *Principles of Analytical Calculation* (Cambridge [England], 1803), page xxvii, where we read, "Again $\overset{\cdots}{xy}$ or $(xy)^{\cdots}$ is not so convenient as $d^3(xy)$," and in a book of Prändel[2] where the English manner of denoting a fluxion is given as $x^{\cdot}$ and $y^{\cdot}$. This displacement of the dot is found in articles contributed by Elizur Wright, of Tallmadge, Ohio, to the *American Journal of Science and Arts* for the years 1828, 1833, 1834.

In 1801 Samuel Webber, then professor of mathematics and later president of Harvard College, published his *Mathematics Compiled from the Best Authors.* In the second volume he touched upon fluxions and used the Newtonian dots. This notation occurred also in the *Transactions of the American Philosophical Society,* of Philadelphia, in an article by Joseph Clay who wrote in 1802 on the "Figure of the Earth." It is found also in the second volume of Charles Hutton's *Course of Mathematics,* American editions of which appeared in 1812, 1816(?), 1818, 1822, 1828, and 1831.

An American edition of S. Vince's *Principles of Fluxions* appeared in Philadelphia in 1812. That early attention was given to the study of fluxions at Harvard College is shown by the fact that in the interval 1796–1817 there were deposited in the college library twenty-one mathematical theses which indicate by their titles the use of fluxions.[3] These were written by members of Junior and Senior classes. The last thesis referring in the title to fluxions is for the year 1832.

At West Point, during the first few years of its existence, neither fluxions nor calculus received much attention. As late as 1816 it is stated in the West Point curriculum that fluxions were "to be taught

[1] William L. Kingsley, *Yale College; a Sketch of Its History,* Vol. II, p. 497, 498.

[2] J. G. Prändel, *Kugldreyeckslehre und höhere Mathematik* (München, 1793), p. 197.

[3] Justin Winsor, *Bibliographical Contributions,* No. 32. Issued by the Library of Harvard University (Cambridge, 1888).

at the option of professor and student." In 1817, Claude Crozet, trained at the Polytechnic School in Paris, became teacher of engineering. A few times, at least, he used in print the Newtonian notation, as, for instance, in the solution, written in French, of a problem which he published in the *Portico*, of Baltimore, in 1817. Robert Adrain, later professor in Columbia College and also at the University of Pennsylvania, used the English notation in his earlier writings, for example, in the third volume of the *Portico*, but in Nash's *Ladies and Gentlemen's Diary*, No. II, published in New York in 1820, he employs the dx.

Perhaps the latest regular use of the dot notation in the United States is found in 1836 in a text of B. Bridge.[1]

In the *American Journal of Science and Arts* the earliest paper giving the Leibnizian symbols was prepared by A. M. Fisher, of Yale College. It is dated August, 1818, and was printed in Volume V (1822). The earliest articles in the *Memoirs of the American Academy of Arts and Sciences* which contain the "d-istic" signs bear the date 1818 and were from the pen of F. T. Schubert and Nathaniel Bowditch.[2] In these *Memoirs*, Theodore Strong, of Rutgers College, began to use the Leibnizian symbolism in 1829. But the publication which placed the Leibnizian calculus and notation within the reach of all and which marks the time of the real beginning of their general use in the United States was the translation from the French of Bézout's *First Principles of the Differential and Integral Calculus*, made by John Farrar, of Harvard University, in 1824. Between 1823 and 1831 five mathematical theses were prepared at Harvard, which contained in their titles the words "Differential Calculus" or "Integral Calculus."[3] After 1824 French influences dominated all instruction in higher analysis at Harvard.

7. Symbols for Passing to the Limit

631. *S. Lhuilier.*—The abbreviation "lim." for "limit" does not seem to have been used before Simon Lhuilier,[4] who writes "lim. $q:Q$"

[1] B. Bridge, *Treatise on . . . . the three Conic Sections*. From the second London edition (New Haven, 1836), p. 75.

[2] *Memoirs of the American Academy of Arts and Sciences*, Vol. IV, Part I (Cambridge, Mass., 1818), p. 5, 47.

[3] Justin Winsor, *Bibliographical Contributions*, No. 32. Issued by the Library of Harvard University (Cambridge, 1888).

[4] Lhuilier, *Exposition élémentaire des principes des calculs supérieurs* (Berlin, 1786), p. 24, 31.

and "Lim. $\frac{\Delta y}{\Delta x}$." It was also used by Garçaõ Stockler, of Lisbon, in 1794; by L. N. M. Carnot[1] (who used also L); and by Brinkley,[2] of Dublin.

A. L. Cauchy wrote "lim." and pointed out[3] that "lim. (sin. x)" has a unique value 0, while "lim. $\left(\left(\frac{1}{x}\right)\right)$" admits of two values and "lim. $\left(\left(\sin.\ \frac{1}{x}\right)\right)$" of an infinity of values, the double parentheses being used to designate all the values that the inclosed function may take, as x approaches zero. In limit and remainder theorems Cauchy designated[4] by θ an undetermined value lying between 0 and 1 for representing a value $x+\theta h$ situated between x and $x+h$. Some writers employ the form ϑ of the Greek letter. The period in "lim." was gradually dropped, and "lim" came to be the recognized form.

632. *E. H. Dirksen*[5] in Berlin used the abbreviation *Gr.* of the German word *Grenze*, in place of the usual contraction of the Latin *limes*. Dirksen writes "$\overset{m=\infty}{Gr.}\ a_m=(+, -, E, \infty, 0)$" to indicate that the limit of a certain infinite series is "positiv und negativ, endlich, unendlich und Null."

633. *The Bolyais.*—About the same time[6] Wolfgang and Johann Bolyai wrote "$x \overset{\cdot}{\frown} a$" to denote x approaching the limit a ("x tendere ad limitem a").

634. *K. Weierstrass.*—In the nineteenth century many writers felt the need of a symbolism indicating also the limit which the independent variable approaches. The notation which has been used widely in Europe is "$\mathrm{Lim}_{x=a}$," to express "the limit as x approaches a." It is found in papers of Weierstrass, who in 1841[7] wrote "lim"

[1] *Œuvres mathématique du Citoyen Carnot* (à Basle, 1797), p. 168, 191–98.

[2] John Brinkley in *Trans. Royal Irish Academy*, Vol. XIII (Dublin, 1818), p. 30.

[3] A. L. Cauchy, *Cours d'analyse* (Paris, 1821), p. 13, 14.

[4] Cauchy, *Exercices de mathématiques* (Paris, 1826); *Œuvres complètes* (2d ser.), Vol. VI, p. 24.

[5] E. H. Dirksen, *Abhandlungen d. K. P. Akademie d. Wissensch.* (Berlin, 1832), Th. I, p. 77–107.

[6] Wolfgangi Bolyai de Bolya, *Tentamen* (2d ed.), Tome II (Budapestini, 1904), p. xix, 361. The first edition appeared in 1832.

[7] K. Weierstrass, *Mathematische Werke*, Band I (Berlin, 1894), p. 60.

and in 1854[1] "$\underset{n=\infty}{\text{Lim.}}\, p_n = \infty$." W. R. Hamilton in 1853[2] used "$\underset{n=\infty}{\lim} \{ \; \}$." Familiar among admirers of Weierstrassian rigor are the expressions "ϵ method of proof" and "ϵ-definition," Weierstrass having begun in his early papers[3] to use ϵ in his arithmetized treatment of limits. The epsilon was similarly used by Cauchy[4] in 1821 and later, but sometimes he wrote[5] δ instead. Cauchy's δ is sometimes associated with Weierstrass' ϵ in phrases like "ϵ and δ methods" of demonstration. Klein[6] chose η in place of ϵ.

635. *J. E. Oliver.*—In the United States the symbol $\doteq$ has been used extensively. It seems to have been introduced independently of European symbolisms. Wolfgang Bolyai[7] used $\doteq$ to express "absolute equality." We have found one Austrian author, A. Steinhauser, using it for "nearly equal," before it was employed in the United States. Letting "Num. log." signify the antilogarithm, Steinhauser[8] writes "Num. log. $0\cdot015682 = 1\cdot03676|\underline{8} \doteq 1\cdot03677$, wobei das Zeichen $\doteq$ soviel als nahezu gleich bedeutet." In America the symbol $\doteq$ is due to James E. Oliver, of Cornell University,[9] and appeared in print in 1880 in W. E. Byerly's[10] *Calculus*. This symbol was used when it became desirable to indicate that the variable was not allowed to reach its limit, though in recent years it has been used, also when no such restriction seemed intended. Oliver[11] himself used it in print in 1888. He says: "Using $\doteq$ in the sense 'Approaches in value to, as a limit,' $U \doteq_{\Delta x} C$ would naturally mean 'U approaches C as a

[1] *Crelle's Journal*, Vol LI (1856), p. 27; *Mathematische Werke*, Band I, p. 155.

[2] W. R. Hamilton, *Lectures on Quaternions* (Dublin, 1853), p. 569.

[3] K. Weierstrass, *op. cit.*, Vol. I, p. 56.

[4] Cauchy, *Cours d'analyse* (1821), p. 49, 50, 61, etc.

[5] Cauchy, *Résumé des leçons sur le calcul infinitésimal* in *Œuvres* (2d ser.), Vol. IV, p. 149.

[6] Felix Klein, *Anwendung der Differential- und Integralrechnung auf Geometrie* (Leipzig, 1907), p. 60.

[7] Wolfgangi Bolyai de Bolya, *Tentamen* (2d ed.), Vol. I (Budapest, 1897), p. xi, 214.

[8] Anton Steinhauser, *Lehrbuch der Mathematik*, "Algebra" (Wien, 1875). p. 292.

[9] See D. A. Murray, *An Elementary Course in the Integral Calculus* (New York, 1898), p. 4 n.

[10] W. E. Byerly, *Elements of the Differential Calculus* (Boston, 1880), p. 7.

[11] *Annals of Mathematics* (Charlottesville, Va.), Vol. IV (1888), p. 187, 188. The symbol is used also in Oliver, Wait, and Jones's *Algebra* (Ithaca, 1887), p. 129, 161.

limit, when Δx approaches its limit (viz. 0).' " Again he says: "Between its meaning as first used by me, 'Is nearly equal to,' and Byerly's modification, 'Approaches as a limit,' and perhaps yet other useful shades of interpretation, the context will sufficiently determine." To mark the degree of approximation, Oliver writes $\stackrel{1}{=}, \stackrel{2}{=}, \ldots, \stackrel{n}{=}$, so that "$U \stackrel{n}{=}_h V$" would mean that $U-V$ or $(U-V)/\sqrt{UV}$ is of at least the nth order of smallness or negligibility, h being of the first order. James Pierpont[1] writes $a_n \doteq l$ for $\lim a_n = l$.

W. H. Echols,[2] of the University of Virginia, advocated the use of the symbol £ for the term "limit" and the symbol $(=)$ in preference to $\doteq$. E. W. Hobson[3] uses the sign $\backsim$ to indicate approach to a limit.

636. *J. G. Leathem.*—In England an arrow has been used in recent years in place of $\doteq$. In 1905 J. G. Leathem,[4] of St. John's College, Cambridge, introduced $\longrightarrow$ to indicate continuous passage to a limit, and he suggested later (1912) that a dotted arrow might appropriately represent a saltatory approach to a limit.[5] The full arrow is meeting with general adoption nearly everywhere. G. H. Hardy[6] writes $\lim\limits_{n\rightarrow\infty} (1/n) = 0$ and introduces the $\longrightarrow$ with the remark, "I have followed Mr. J. G. Leathem and Mr. T. J. I'A. Bromwich in always writing $\lim\limits_{n\rightarrow\infty}$, $\lim\limits_{x\rightarrow\infty}$, $\lim\limits_{x\rightarrow a}$, and not $\lim\limits_{n=\infty}$, $\lim\limits_{x=\infty}$, $\lim\limits_{x=a}$. This

[1] James Pierpont, *Theory of Functions of Real Variables* (New York), Vol. I (1905), p. 25.

[2] W. H. Echols, *Elementary Text-Book on the Differential and Integral Calculus* (New York, 1902), Preface, also p. 5. We quote from the Preface: ". . . . The use of the 'English pound' mark for the symbol of 'passing to the limit' is so suggestive and characteristic that its convenience has induced me to employ it in the text, particularly as it has been frequently used for this purpose here and there in the mathematical journals. The use of the 'parenthetical equality' sign $(=)$ to mean 'converging to' has appeared more convenient . . . . than the dotted equality $\doteq$, which has sometimes been used in American texts."

[3] E. W. Hobson, *Theory of Functions of a Real Variable* (2d ed.; Cambridge, 1921).

[4] J. G. Leathem, *Volume and Surface Integrals Used in Physics* (Cambridge, 1905).

[5] The arrow was used by Riemann in a lecture of 1856–57 in a manner which suggests limits. He says (*Gesammelte Math. Werke, Nachträge* [Leipzig, 1902], p. 67): "Ist a ein Verzweigungspunkt der Lösung einer linearen Differentialgleichung zweiter Ordnung und geht, während x sich im positiven Sinn um a bewegt, $\mathfrak{z}_1$ über in $\mathfrak{z}_3$ und $\mathfrak{z}_2$ in $\mathfrak{z}_4$, was kurz durch $\mathfrak{z}_1 \rightarrow \mathfrak{z}_3$ und $\mathfrak{z}_2 \rightarrow \mathfrak{z}_4$ angedeutet werden soll, so ist $\mathfrak{z}_3 = t\mathfrak{z}_1 + u\mathfrak{z}_2$, $\mathfrak{z}_4 = r\mathfrak{z}_1 + s\mathfrak{z}_2$. Ist ϵ irgend eine Konstante, so ist $\mathfrak{z}_1 + \epsilon\mathfrak{z}_2 \rightarrow \mathfrak{z}_3 + \epsilon\mathfrak{z}_4$."

[6] G. H. Hardy, *Course of Pure Mathematics* (1908), p. 116; see also the Preface.

change seems to me one of considerable importance, especially when '∞' is the 'limiting value.' I believe that to write $n=\infty$, $x=\infty$ (as if anything ever were 'equal to infinity'), however convenient it may be at a later stage, is in the early stages of mathematical training to go out of one's way to encourage incoherence and confusion of thought concerning the fundamental ideas of analysis." Sometimes the "lim" is omitted,[1] as in $\frac{\Delta u}{\Delta x}\rightarrow u'$. F. S. Carey,[2] following a suggestion made by James Mercer, uses for monotone sequences an arrow with a single barb, either upper or lower; thus, $\rightharpoondown$ "tends up to" the limit, $\rightharpoonup$ "tends down to" the limit. Leathem[3] used sloped arrows for this purpose.

637. *G. L. Dirichlet.*—G. Lejeune Dirichlet[4] in 1837 introduced a symbolism which we give in his own words: "Es wird im Folgenden nöthig sein, diese beiden Werthe von $\varphi(\beta)$ zu unterscheiden, und wir werden sie durch $\varphi(\beta-0)$ und $\varphi(\beta+0)$ bezeichnen." We have here the first suggestion of our present notation, "lim $x=a+0$," when x converges to a by taking only values greater than a, and "lim $x=a-0$," when x converges to a by taking values all less than a.

A. Pringsheim.—The upper and the lower limit of y as a function of x was designated by M. Pasch,[5] of Giessen, as "$\lim\limits_{v=\infty}\sup y$" and "$\lim\limits_{v=\infty}\inf y$." For the same purpose A. Pringsheim, of Munich, introduced the notations $\overline{\lim\limits_{v=+\infty}}\, a_\nu=A$ and $\underline{\lim\limits_{v=+\infty}}\, a_\nu=\alpha$ to denote, respectively, the upper and lower limits of the variable a_ν,[6] and $\overline{\underline{\lim\limits_{\nu=+\infty}}}\, a_\nu$ to indicate that one may take at pleasure either the upper or the lower limit.

Paul du Bois-Reymond,[7] of the University of Freiburg in Baden,

[1] See, for example, Maurice Laboureur, *Cours de calcul algébrique différentiel et intégral* (Paris et Liège, 1913), p. 102.

[2] F. S. Carey, *Infinitesimal Calculus* (London, 1919), Preface.

[3] J. G. Leathem, *Mathematical Theory of Limits* (London, 1925), Preface.

[4] *Repertorium der Physik*, Vol. I (Berlin, 1837), p. 170.

[5] M. Pasch, *Mathematische Annalen*, Vol. XXX (1887), p. 134.

[6] *Sitzungsberichte Akad. München, Math.-phys. Classe*, Vol. XXVIII (1898), p. 62. See also *Encyclopédie des scienc. math.*, Tome I, Vol. I, p. 189, n. 224; p. 232, n. 45.

[7] Paul du Bois-Reymond in *Annali di mathematica pura ed applicata*. Diretti da F. Brioschi e L. Cremona (Milano, 2d ser.), Tomo IV (1870–71), p. 339.

introduced the notations $f(x) \succ \varphi(x)$, $f(x) \sim \varphi(x)$, $f(x) \prec \varphi(x)$, as equivalents to the formulae

$$\lim \frac{f(x)}{\varphi(x)} = \infty \ , \qquad \lim \frac{f(x)}{\varphi(x)} \text{ is finite,} \qquad \lim \frac{f(x)}{\varphi(x)} = 0 \ .$$

These symbols were used by A. Pringsheim[1] who in 1890 added $f_1(n) \cong af_2(n)$ for $\lim \frac{f_1(n)}{f_2(n)} = a$. The sign $\simeq$ is sometimes used to mark the lower limit,[2] as in "$J \simeq 45.10^{-40}$."

L. Scheffer.—The four values (viz., Du Bois-Reymond's *Unbestimmtheitsgrenzen*) which the difference-quotient $\frac{\Delta y}{\Delta x}$ approaches, as Δx approaches zero (y being a single-valued function of x), were designated by Dini[3] by the symbols Λ, λ, Λ', λ'; the first two being the right-hand upper and lower limits, the last two the left-hand upper and lower limits. Ludwig Scheffer[4] used in 1884 for these four limits, respectively, the symbols D^+, D_+, D^-, D_-, which were modified by M. Pasch[5] in 1887 to D^+, $_+D$, D^-, $_-D$.

G. Peano.—The countless values when x approaches zero, of lim $\left(\left(\sin \frac{1}{x}\right)\right)$, cited by Cauchy in 1821, constitute a class, for the designation of which G. Peano[6] introduced the sign "Lm," in place of the ordinary "lim." By Lm x Peano means "les valeurs limites de x" or "la classe limite de x." The need of this new abbreviation is an indication that, since the time when Cauchy began to publish, a new concept, the idea of classes, has become prominent in mathematics. By Lm (f, u, x) Peano[7] designates "les valeurs limites de la fonction f, lorsque la variable, en variant dans la classe u, tend vers x."

Peano[8] designates an interval of integration by the use of a capital letter I placed horizontally and in an elevated position; thus $a \vdash\!\dashv b$ means the interval from a to b. He writes also $S^1(f, a \vdash\!\dashv b)$ for *l'intégrale*

[1] A. Pringsheim in *Mathematische Annalen*, Vol. XXXV (1890), p. 302.

[2] A. Eucken, *Zeitschrift der physikalischen Chemie*, Band C, p. 164.

[3] Dini, *Fondamenti per la teorica delle funzioni di variabili reali* (1878), art. 145, p. 190.

[4] L. Scheffer, *Acta mathematica*, Vol. V (Stockholm, 1884), p. 52, 53.

[5] M. Pasch, *Mathematische Annalen*, Vol. XXX (1887), p. 135.

[6] G. Peano, *Formulaire mathématique*, Tome IV (Turin, 1903), p. 148.

[7] *Op. cit.*, p. 164.

[8] G. Peano, *op. cit.*, p. 178, 179.

par excès or the inferior limit of the sums s' of f in the interval from a to b, and $S_1(f, a \overline{\vdash} b)$ for *l'intégrale par défaut* or the superior limit of the sums s_1 of f in the interval a to b. Peano remarks that his notation $S(f, a \overline{\vdash} b)$ has the advantage of being applicable also when the integral does not extend over an interval, but over any class. In an earlier publication[1] Peano used the sign $\vdash$ to mark an interval with the right end open and $\dashv$ one with the left end open. More common notations are[2] $(\alpha\, \beta)$ for the interval from α to β; also $\alpha \leqq \xi \leqq \beta$, $\alpha < \xi \leqq \beta$, $\alpha \leqq \xi < \beta$, $\alpha < \xi < \beta$, the first and last intervals being closed and open, respectively, the others partially open, and α, β finite.

W. H. Young.—The consideration of the plurality of limits has brought forth other special notations. Thus W. H. Young[3] states in 1910: "We use the notation $\underset{x=a}{\mathrm{Llt}}\, f(x)$ to denote the set of all the limits of $f(x)$ at a point a, while, if it is known that there is an unique limit, we write $\underset{x=a}{\mathrm{Lt}}\, f(x)$ for that limit. . . . . If there are two independent variables x and y, the limits at (a, b) . . . . are called *double limits.*" He considers "*repeated limits* of $f(x, y)$ first with respect to x and then with respect to y, and written $\underset{y=b}{\mathrm{Llt}}\, \underset{x=a}{\mathrm{Lt}}\, f(x, y)$." Similarly he considers repeated limits, first with respect to y and then with respect to x.

638. *The sign* $\frac{0}{0}$.—The indeterminate case, zero divided by zero, is treated by L'Hospital in his *Analyse des infiniments petits* (1696), but he does not use the symbol. Johann Bernoulli[4] discusses this indeterminedness in 1704, and uses such symbolism as $\frac{a0}{0}$, $\frac{0a}{0}$, $\frac{0m}{0n}$, but not $\frac{0}{0}$ in its nakedness. However, in 1730 he[5] did write $\frac{0}{0}$. G. Cramer[6] used $\frac{0}{0}$ in a letter to James Stirling, dated February 22, 1732. The symbol occurs repeatedly in D'Alembert's article "Différential" in

[1] G. Peano, *Analisi Infinitesimale*, Vol. I (Turin, 1893), p. 9, 10. See also *Formulario mathematico*, Vol. V (1908), p. 118.

[2] J. Harkness and F. Morley, *Theory of Analytic Functions* (London, 1898), p. 113–15.

[3] W. H. Young, *The Fundamental Theorems of the Differential Calculus* (Cambridge University Press, 1910), p. 3, 4.

[4] Johann Bernoulli in *Acta eruditorum* (1704), p. 376, 379, 380.

[5] Johann Bernoulli, *Opera omnia*, Vol. III, p. 183; *Mémoires de l'acad. r. d. sciences de Paris*, année 1730, p. 78.

[6] Charles Tweedie, *James Stirling; A Sketch of His Life and Works* (Oxford, 1922), p. 127.

Diderot's *Encyclopédie* of 1754. It became the battleground of contending philosophic thought on the calculus. A. T. Bledsoe[1] expresses himself with regard to it as follows: "We encounter $\frac{0}{0}$, the most formidable of all the symbols or enigmas in the differential calculus. . . . . Even Duhamel shrinks from a contact with it, although its adoption seems absolutely necessary to perfect the method of limits. . . . . This symbol is repudiated by Carnot and Lagrange. It is adopted by Euler and D'Alembert; but they do not proceed far before it breaks down under them. It is, nevertheless, one of the strongholds and defences of the method of limits, which cannot be surrendered or abandoned without serious and irreparable loss to the cause. . . . . This singular crusade of mathematicians against one poor symbol $\frac{0}{0}$, while all other symbols of indetermination are spared, is certainly a curious fact."

The sign $\frac{0}{0}$ was introduced quite frequently in textbooks on algebra of the first half of the nineteenth century.[2]

8. Concluding Observations

639. In considering the history of the calculus, the view advanced by Moritz Cantor presses upon the mind of the reader with compelling force. Cantor says: "We have felt that we must place the main emphasis upon the notation. This is in accordance with the opinion which we have expressed repeatedly that, even before Newton and Leibniz, the consideration of infinitesimals had proceeded so far that a suitable notation was essential before any marked progress could be made."[3]

Our survey of calculus notations indicates that this need was met, but met conservatively. There was no attempt to represent all reasoning in the calculus by specialized shorthand signs so as to introduce a distinct sign language which would exclude those of ordinary written or printed words. There was no attempt to restrict the exposition of

[1] Albert Taylor Bledsoe, *The Philosophy of Mathematics* (Philadelphia, 1886 [copyright, 1867]), p. 215, 222.

[2] For example, in C. F. Fournier, *Eléments d'arithmétique et d'algèbre,* Vol. II (Nantes, 1842), p. 217, 270.

[3] M. Cantor, *Vorlesungen über Geschichte der Mathematik,* Vol. III (2d ed.; Leipzig, 1901), p. 167.

theory and application of the calculus to ideographs. Quite the contrary. Symbols were not generally introduced, until their need had become imperative. Witness, for instance, the great hesitancy in the general acceptance of a special notation for the partial derivative.

It is evident that a sign, to be successful, must possess several qualifications: It must suggest clearly and definitely the concept and operation which it is intended to represent; it must possess adaptability to new developments in the science; it must be brief, easy to write, and easy to print. The number of desirable mathematical symbols that are available is small. The letters of the alphabet, the dot, the comma, the stroke, the bar, constitute the main source of supply. In the survey made in this paper, it was noticed that the forms of the fourth letter of the alphabet, d, D, and ∂, were in heavy demand. This arose from the fact that the words "difference," "differential," "derivative," and "derivation" all began with that letter. A whole century passed before any general agreement was reached among mathematicians of different countries on the specific use which should be assigned to each form.

The query naturally arises, Could international committees have expedited the agreement? Perhaps the International Association for the Promotion of Vector Analysis will afford an indication of what may be expected from such agencies.

An interesting feature in our survey is the vitality exhibited by the notation $\frac{dy}{dx}$ for derivatives. Typographically not specially desirable, the symbol nevertheless commands at the present time a wider adoption than any of its many rivals. Foremost among the reasons for this is the flexibility of the symbol, the ease with which one passes from the derivative to the differential by the application of simple algebraic processes, the intuitional suggestion to the mind of the reader of the characteristic right triangle which has dy and dx as the two perpendicular sides. These symbols readily call to mind ideas which reach to the very heart of geometric and mechanical applications of the calculus.

For integration the symbol $\int$ has had practically no rival. It easily adapted itself to the need of marking the limits of integration in definite integrals. When one considers the contributions that Leibniz has made to the notation of the calculus, and of mathematics in general, one beholds in him the most successful and influential builder of symbolic notation that the science ever has had.

FINITE DIFFERENCES

640. *Early notations.*—Brook Taylor explains his notation in the Introduction to his *Method of Increments,*[1] in a passage of which the following is a translation: "Indeterminate quantities I consider here as continually augmented by increments or diminished by decrements. The integral indeterminates I designate by the letters z, x, v, etc., while their increments, or the parts then added, I mark by the same letters dotted below, $\underset{\cdot}{z}$, $\underset{\cdot}{x}$, $\underset{\cdot}{v}$, etc. I represent the increments of the increments, or the second increments of the integrals themselves, by the same letters twice dotted, $\underset{\cdot\cdot}{z}$, $\underset{\cdot\cdot}{x}$, $\underset{\cdot\cdot}{v}$, etc. I designate the third increments by the letters triply dotted, $\underset{\cdot\cdot\cdot}{z}$, $\underset{\cdot\cdot\cdot}{x}$, $\underset{\cdot\cdot\cdot}{v}$, etc., and so on. Indeed, for greater generality I sometimes write characters standing for the number of points: Thus, if n is 3, I designate $\underset{\cdot\cdot\cdot}{x}$ by $\underset{n}{x}$ or $\underset{3}{x}$; if $n=0$, I designate the Integral itself x by $\underset{n}{x}$ or $\underset{0}{x}$; if n is -1, I designate the quantity of which the first increment is x by $\underset{n}{x}$ or $\underset{-1}{x}$; and so on. Often in this Treatise I mark the successive values of one and the same variable quantity by the same letter marked by lines; namely, by designating the present value by a simple letter, the preceding ones by superscribed grave accents, and the subsequent ones by strokes written below. Thus for example, $\grave{\grave{x}}$, $\grave{x}$, x, $x_{\backprime}$, $x_{\backprime\backprime}$ are five successive values of the same quantity, of which x is the present value, $\grave{x}$ and $\grave{\grave{x}}$ are preceding values, and $x_{\backprime}$ and $x_{\backprime\backprime}$ are subsequent values." Taylor uses at the same time the Newtonian symbols for fluxions and fluents. Two years later Taylor introduced a symbol for summation. He says:[2] "When Occasion requires that a variable Quantity, suppose x, is to be look'd upon as an Increment, I denote its Integral by the Letter included between two Hooks [ ]. Also the Integral of the Integral $[x]$ or the second Integral of x, I denote by putting the Number 2 over the first of the Hooks, as $\overset{2}{[}x]$, ... so that it is $\overset{2}{[}x]=\overset{3}{[}\underset{\cdot}{x}]$, $[x]=\overset{2}{[}\underset{\cdot}{x}]$, $x=[\underset{\cdot}{x}]$."

It was not long before alterations to Taylor's notation were made in England. In E. Stone's *New Mathematical Dictionary* (London, 1726), article "Series," preference is given to the modified symbols

[1] Brook Taylor, *Methodus Incrementorum directa et inversa* (London, 1715), p. 1.

[2] Brook Taylor in *Philosophical Transactions*, No. 353 (London, 1817), p. 676; *ibid.* (3d abridged ed., Henry Jones), Vol. IV (London, 1749), p. 130.

introduced by Samuel Cunn, a textbook writer and, with Raphson, a translator into English of Newton's *Arithmetica universalis*. Letting q be a constant increment, Cunn marks $Q+q$ by $\underset{\prime}{Q}$, $Q+2q$ by $\underset{\prime\prime}{Q}$, $Q+3q=\underset{\prime\prime\prime}{Q}$, and so on; also $Q-q$ by $\grave{Q}$, $Q-2q$ by $\ddot{\grave{Q}}$, etc. Stone remarks that Taylor and others chose to denote the increments by the same letters with the integrals and only for the sake of distinction printed them beneath. Comparing the two notations, we have $q=\underset{\cdot}{Q}$, $n+\underset{\cdot}{n}=\underset{\prime}{n}$.

Taylor's notation, with some modifications, was employed by W. Emerson.[1] Any letters s, t, v, x, y, z, etc., are put for "integral quantities," the different values of any such quantity are denoted by the same letter with numeral subscripts. Emerson continues: "If Z be an integral, then z, $\underset{1}{z}$, $\underset{2}{z}$, $\underset{3}{z}$, etc. are the present value, and the first, second, third, etc. successive values thereof; and the preceding values are denoted with figures and negative signs; thus $\underset{-1}{z}$, $\underset{-2}{z}$, $\underset{-3}{z}$, $\underset{-4}{z}$, are the first, second, third, fourth preceding values. . . . . The increments are denoted with the same letters and points under them, thus $\underset{\cdot}{x}$ is the increment of x, and $\underset{\cdot}{z}$ is the increment of z. Also $\underset{1\cdot}{x}$ is the increment of $\underset{1}{x}$; and $\underset{n\cdot}{x}$ of $\underset{n}{x}$, etc. . . . . If x be any increment, then $[x]$ denotes the integral of x, and $\overset{2}{[x]}$ denotes the integral of $[x]$. . . . ."

Taylor's dot notation for differences prevailed with Edward Waring[2] who, using it along side of the fluxional notation, writes, for instance, "incrementum logarithmi quantitatis (x), cujus fluxio est $\frac{\dot{x}}{x}$, erit log. $(x+\underset{\cdot}{x})-$log. $(x)=$log. $\frac{x+\underset{\cdot}{x}}{x}$, cujus fluxio erit $-\frac{\underset{\cdot}{x}\dot{x}}{x(x+\underset{\cdot}{x})}$ . . . . ."

On the European Continent, Leibniz used the sign d both for finite and infinitesimal differences; he employed $\int$ for the sum of terms in a series as well as for integration. For example, he says:[3] "Supposing that $1+1+1+1+1+$ etc. $=x$ or that x represents the natural numbers, for which $dx=1$, then $1+3+6+10+$etc. $=\int x$, $1+4+10+20+$ etc. $=\int\int x$."

[1] William Emerson, *Method of Increments* (London, 1763), p. 2.

[2] Edward Waring, *Meditationes analyticae* (2d ed.; Cambridge, 1785), p. 307.

[3] *Historia et origo calculi differentialis* in *Leibnizens Mathematische Schriften*, Vol. V (1858), p. 397.

Wide acceptance has been given to the symbolism on finite differences introduced by L. Euler.[1] In place of Leibniz' small Latin letter d, he used the corresponding Greek capital letter Δ which Johann Bernoulli had used previously for differential coefficient (§§ 539, 572, 596). In *Institutiones calculi differentialis*, Euler begins in the first chapter by letting ω stand for the increment of a variable x, but soon after enters upon a more elaborate notation. Letting y, y^I, y^{II}, y^{III}, etc., be the values of y corresponding to the x-values x, $x+\omega$, $x+2\omega$, $x+3\omega$, etc., he puts $y^I-y=\Delta y$, $y^{II}-y^I=\Delta y^I$, $y^{III}-y^{II}=\Delta y^{II}$, etc. Thus Δy expresses the increment which the function y assumes, when in place of x one writes $x+\omega$. Proceeding further, he lets $\Delta\Delta y=\Delta y^I-\Delta y$, $\Delta\Delta y^I=\Delta y^{II}-\Delta y^I$, etc., and calls $\Delta\Delta y$ the "differentia secunda ipsius y; $\Delta\Delta y^I$ differentia secunda ipsius y^I," etc., and $\Delta^3 y$, $\Delta^3 y^I$ the "differentiae tertiae," $\Delta^4 y$, $\Delta^4 y^I$, "differentiae quartae," and so on. He represents the values of y corresponding to x, $x-\omega$, $x-2\omega$, etc., by y, y_I, y_{II}, etc., so that $y=\Delta y_I+y_I$, etc., and $y=\Delta y_I+\Delta y_{II}+\Delta y_{III}$, etc. It is here that he introduces also the letter Σ for "sum"; he says: "Quemadmodum ad differentiam denotandam vsi sumus signo Δ, ita summam indicabimus signo Σ." The general features of Euler's notations were adopted by Lagrange and Laplace in some of their papers. Lagrange[2] in 1772 assigns to the variables x, y, z . . . . the *accroissements* ξ, ψ, ζ, ... and to the function u the *accroissement* Δu, and the differences of higher order $\Delta^2 u$, $\Delta^3 u$, ... , $\Delta^\lambda u$. He supposes that λ may be negative and writes $\Delta^{-1}=\Sigma$, $\Delta^{-2}=\Sigma^2$, ... In 1792 Lagrange[3] lets T_0, T_1, T_2, ... , T_n, T_{n+1}, ... be the successive terms of a series D_1, D_2, D_3, the successive differences of the terms, i.e., $D_1=T_1-T_0$, $D_2=T_2-2T_1+T_0$, etc. He writes $D_m=(T_1-T_0)_m$, and by the analogy existing between exponents and the indices, he expands the binomial $(T_1-T_0)_m$. By making m negative the differences change into sums, which he represents by S_1, S_2, S_3, ... so that $S_1=D_{-1}$, $S_2=D_{-2}$, ... When interpolating between successive terms of T_0, T_1, T_2, ... he marks the terms interpolated $T_{\frac{1}{2}}$, $T_{\frac{3}{2}}$, $T_{\frac{5}{2}}$, ... When dealing simultaneously with two series, he employs for the second series the corresponding small letters t_0, t_1, t_2, ... , d_0, d_1, d_2, ... Lagrange considers also double and triple series, in which the

[1] Leonhard Euler, *Institutiones calculi differentialis* (Petersburgh, 1755), p. 3–7, 27.

[2] *Œuvres de Lagrange*, Vol. III (Paris, 1869), p. 443, 450, 451; *Nouveaux mémoires de l'acad. r. d. scienc. et bell.-lett. de Berlin*, année 1772.

[3] *Œuvres de Lagrange*, Vol. V (Paris, 1870), p. 667, 668, 673, 679; *Nouveaux mémoires de l'acad. r. d. scienc. et bell.-lett. de Berlin*, années 1792 et 1793.

terms vary in two or more different ways, and introduces symbols

$$\begin{array}{llll} T_{0,0} & T_{1,0} & T_{2,0} & \ldots \\ T_{0,1} & T_{1,1} & T_{2,1} & \ldots \\ T_{0,2} & T_{1,2} & T_{2,2} & \ldots \\ . & . & . & . \end{array}$$

and, correspondingly, $D_{0,0}$, $D_{1,0}$, ... , $D_{0,1}$, $D_{1,1}$, ... , $D_{0,2}$, $D_{1,2}$, ... , or generally $D_{m,n}$ for the successive differences.

Laplace[1] lets the variables t, t_1, t_2, ... increase respectively by a, a_1, a_2, ... , and lets u' be the value of a function of these variables after they have received their increment. He writes $u'-u=\Delta u$, and uses also the symbols $\Delta^2 u$, $\Delta^i u$, Σ, and Σ^i.

641. *Later notations.*—The symbols d, D, Δ, $\int$, Σ were used by Arbogast,[2] Français,[3] and others in the development of functions in series and in treatment of recurrent series. Kramp's substitution of the German letters $\mathfrak{D}$ and $\mathfrak{S}$ for the Greek Δ and Σ did not meet with wide acceptance.[4] These symbols were used sometimes as symbols of operation (d and D in differentiation, Δ for finite differences, $\int$ for integration, Σ for finite summation), and sometimes as if they were veritable algebraic quantities. Servois[5] studied the laws of the calculus with symbols and showed that Δ, D, Σ, $\int$ possessed the *propriété commutative, propriété distributive,* expressions which have secured permanent adoption. Later authors have used a great variety of symbolism in the calculus of operations, of which the more important are due to Boole,[6] Carmichael,[7] and Casorati.[8]

Euler's notation found entrance into England at the time of the

[1] *Œuvres de Laplace*, Vol. IX (Paris, 1893), p. 316, 318; *Mémoires de l'acad. r. d. scienc. de Paris,* année 1777 (Paris, 1780).

[2] L. F. A. Arbogast, *Du calcul des dérivations* (Strasbourg, an VIII [1800]), p. 1–159, 230. See A. Cayley in *Philosophical Transactions* (London), Vol. CLI (1861), p. 37. See also *Encyclopédie des scien. math.*, Tom. II, Vol. V, p. 4, 5 (1912).

[3] J. F. Français in *Annales math. pures appl.* (Gergonne), Vol. III (1812–13), p. 244–72. He uses the rounded letter ∂ in place of d.

[4] C. Kramp, *Élémens d'arithmétique universelle* (Cologne, 1808), "Notations."

[5] F. J. Servois, *Annales math. pures appl.* (Gergonne), Vol. V (1814–15), p. 93.

[6] G. Boole, *Philosophical Transactions,* Vol. CXXXIV (London, 1844), p. 225; *Math. Analysis of Logic* (Cambridge, 1847), p. 15–19. *Treatise on Differential Equations* (Cambridge, 1859), p. 371–401.

[7] R. Carmichael, *Treatise on the Calculus of Operations* (London, 1855).

[8] F. Casorati in *Annali di mat. pura ed appl.* (2d ser.), Vol. X (1880–82), p. 10, etc.

introduction of the Leibnizian notation of the calculus, in the early part of the nineteenth century. Euler's symbols are used by J. F. W. Herschel[1] and later by George Boole.[2] Boole says (p. 16, 17): "In addition to the symbol Δ, we shall introduce a symbol E to denote the operation of giving to x in a proposed subject function the increment unity;—its definition being $E_{u_x}=u_{x+1}$." "The two symbols are connected by the equation $E=1+\Delta$."

SYMBOLS IN THE THEORY OF FUNCTIONS

A. SYMBOLS FOR FUNCTIONS IN GENERAL

642. In 1694 Johann Bernoulli[3] represented by the letter n any function whatever of z ("posito n esse quantitatem quomodocunque formatam ex indeterminatis et constantibus") and then proceeded to consider the integral of ndz. In an article[4] of the following year, Jakob (James) Bernoulli proposed a problem in which he lets p and q be any two functions whatever of x ("... p et q quantitates utcunque datas per x denotant"). In 1697 Johann Bernoulli[5] let the capital letter X and also the corresponding Greek letter ξ represent functions of x ("per ξ intelligo quantitatem datam per x et constantes"; "X quantitati itidem ex x et constantibus compositae"). He writes also "$\xi X=$quantitati pure dependenti ab x et constantibus." In a letter to Leibniz he said:[6] "For denoting any function of a variable quantity x, I rather prefer to use the capital letter having the same name X or the Greek ξ, for it appears at once of what variable it is a function; this relieves the memory." To this Leibniz replied:[7] "You do excellently in letting the mark of a function indicate of what letter it is the

[1] J. F. W. Herschel, *Calculus of Finite Differences* (Cambridge, 1820), p. 1, 4.

[2] George Boole, *Calculus of Finite Differences* (3d ed.; London, 1880), p. 2, 4, 5. [Preface to 1st ed. is dated April 18, 1860.]

[3] Johann Bernoulli in *Acta eruditorum* (Leipzig, 1694), p. 437. I am indebted for this reference and a few others to G. Eneström's article in *Bibliotheca mathematica* (2d ser.), Vol. V (1891), p. 89.

[4] Jakob Bernoulli in *Acta eruditorum* (1695), p. 553.

[5] Johann Bernoulli, in *Acta eruditorum* (1697), p. 115.

[6] *Leibnizens Mathematische Schriften,* Vol. III (1856), p. 531. Letter dated Aug. 16/26, 1698.

[7] Letter of Leibniz to Johann Bernoulli, n.d., in *Leibnizens Mathem. Schriften,* Vol. III (1856), p. 537. It should be stated that Leibniz' letter, as printed in the *Commercium Philosophicum et Mathematicum* of G. W. Leibniz and Johann Bernoulli, Vol. I (1745), p. 399, has some slight variations in the symbolism; thus, instead of $\overline{x}\lfloor r.1\rfloor$ we find $\overline{x}\lfloor r:1\rfloor$. Here, and in the other cases given in the letter, a colon is printed where in Gerhardt's edition there is simply a dot.

function, for example, that ξ be a function of x. If there are many functions of that same x, they may be distinguished by numbers. Occasionally I mark the sign of relationship in this manner $\overline{x}\underline{|1|}$, $\overline{x}\underline{|2|}$, etc., that is, any expression in x whatever; again, if one is formed from many, as from x and y, I write $\overline{x;y}\underline{|1|}$, $\overline{x;y}\underline{|2|}$. And when the form is rational, I add r, as in $\overline{x}\underline{|r.1|}$ and $\overline{x}\underline{|r.2|}$ or $\overline{x;y}\underline{|r.1|}$, $\overline{x;y}\underline{|r.2|}$. If the form is rational integral, I write $\overline{x}\underline{|ri.1|}$, $\overline{x}\underline{|ri.2|}$. But in the case of only one function, or only a few of them, the Greek letters suffice, or some such, as you are using." This notation to indicate functions Leibniz never had printed, nor did he refer to it again in his letters to Johann Bernoulli or to other correspondents. On the other hand, the notation proposed by Johann Bernoulli was used by his brother Jakob Bernoulli[1] in 1701, who without use of the word "function" defined B, F, G, C as functions, respectively of b, f, g, c by equations such as $adF=hdf$ or $F=\sqrt{}(aa+ff)$, where a is a constant. This mode of representation was employed extensively during the eighteenth century and not infrequently has been found convenient also in more recent times.

643. The use of a special symbol to stand for a function of a given variable is found again in a memoir[2] of Johann Bernoulli of 1718, where the Greek φ is employed for this purpose. He stated: "en prenant aussi φ pour la caractéristique de ces fonctions." He also writes (p. 246) the differential equation $dy:dz=(\varphi x\pm c):a$, and further on (p. 250), $dy:dx=(\varphi z\pm c):a$. It is to be observed that neither Johann Bernoulli nor his successors Clairaut and D'Alembert in their earlier writings inclose the variable or argument in parentheses.[3] The use of parentheses for this purpose occurs in Euler[4] in 1734, who says, "Si $f\left(\frac{x}{a}+c\right)$ denotet functionem quamcunque ipsius $\frac{x}{a}+c$." This is the first appearance of f for "function." About the same time Clairaut designates a function of x by Πx, Φx, or Δx, without using parentheses.[5] In

[1] Jakob Bernoulli in *Analysin magni problematis isoperimetrici* (Basel, 1701); Jacobi Bernoulli, *Opera*, Vol. II (Geneva, 1744), p. 907, 909.

[2] *Mémoires d. l'acad. d. sciences de Paris* (1718), p. 108; *Opera* (Lausanne et Genève, 1742), Vol. II, p. 243, 246, 250.

[3] G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. VI (1905), p. 111.

[4] L. Euler in *Comment. Petropol. ad annos 1734–1735*, Vol. VII (1840), p. 186, 187, second paging; reference taken from J. Tropfke, *op. cit.*, Vol. II (2d ed., 1921), p. 35.

[5] A. C. Clairaut in *Histoire d. l'acad. d. sciences*, année 1734 (Paris, 1736), *Mém.*, p. 197.

1747 D'Alembert[1] marked functions of the two variables u and s by the signs $\Delta u,s$, $\Gamma u,s$. In 1754 he says:[2] "Soit $\varphi(z)$ une fonction de la variable z, $dz\Delta(z)$ la différence de cette fonction; $dz\Gamma(z)$ la différence de $\Delta(z)$, $dz\Psi(z)$ la différence de $\Gamma(z)$ etc. Soit ensuite $\varphi(z+\zeta)$ une fonction de $z+\zeta$ pareille à la fonction $\varphi(z)$, ζ étant une très-petite quantité. ..."

In 1753 Euler[3] designated a function of x and t by $\Phi:(x, t)$. A year later he wrote:[4] "Elle sera donc une certaine fonction de a et n, que J'indiquerai par $f:(a, n)$, dont la composition nous est encore inconnüe." In 1790 Mascheroni[5] marked functions of the variable u thus, F, u and f, u.

644. A great impetus toward the more general use of the letters f, F, φ and ψ for the designation of functional relations was given by Lagrange[6] in his *Théorie des fonctions analytiques* (Paris, 1797). But such designations occur much earlier in his writings; nor are they the only symbols by which he represented functions. In one of his papers of 1759 "X et Z expriment des fonctions quelconques de la variable x."[7] In another paper of the same year he[8] wrote the equation $y=\Psi(t\sqrt{c}+x)+\Gamma(t\sqrt{c}-x)$, "$\Psi$ et Γ exprimant des fonctions quelconques des quantités $t\sqrt{c}+x$ et $t\sqrt{c}-x$." The year following he wrote[9] "$\varphi(t-ax)$." Somewhat later he[10] discussed a differential equation, "P étant une fonction quelconque de p et Q une fonction quelconque de q," and in another article considered the variation of the function

[1] D'Alembert, *Réflexions sur la cause générale des vents* (Paris, 1747), p. 164. See G. Eneström, *Bibliotheca mathematica* (2d ser., Vol. V 1891), p. 90.

[2] D'Alembert, *Recherches sur differens points importans du système du monde*, Vol. I (Paris, 1754), p. 50.

[3] L. Euler in *Mémoires d. l'acad. d. sciences et des belles lettres* (Berlin, 1753), p. 204.

[4] L. Euler, *op. cit.*, année 1754 (Berlin, 1756), p. 209.

[5] L. Mascheroni, *Adnotationes ad Calculum integralem Euleri* (Ticini, 1790). Reprinted in L. Euler's *Opera omnia* (1st ser.), Vol. XII (1914), p. 484.

[6] *Œuvres de Lagrange*, Vol. IX (Paris, 1881), p. 21, 43, 93, 99, 113, 171, 243, 383.

[7] *Miscellanea Taurinensia*, Vol. I (1759); *Œuvres de Lagrange*, Vol. I (Paris, 1867), p. 23.

[8] *Miscellanea Taurinensia*, Vol. I (1759); *Œuvres de Lagrange*, Vol. I (Paris, 1867), p. 64.

[9] *Miscellanea Taurinensia*, Vol. II (1760–61); *Œuvres*, Vol. I, p. 154.

[10] *Miscellanea Taurinensia*, Vol. IV (1766–69); *Œuvres*, Vol. II (Paris, 1868), p. 20.

φ, and took[1] Φ as a function $\varphi, x, y, z, \ldots$. In 1772 Lagrange[2] took u as a function of x. Another time he[3] wrote Clairaut's equation, $y-px+f(p)=0$, "$f(p)$ dénotant une fonction quelconque de p seul," and he gave $f'(p)=\frac{df(p)}{dp}$. A curious symbol, $\textcircled{I}x$, for our $f(x)$ was used by W. Bolyai.[4]

645. During the early part of the nineteenth century functional notations found their way into elementary textbooks; for instance, into Legendre's *Elements of Geometry*,[5] where a function of p and q was marked $\Phi:(p, q)$.

J. F. W. Herschel[6] uses the sign $f(x)$ and says, "In general $f(f(x))$ or ffx may be written $f^2(x)$ . . . . and $f^m f^n(x)=f^{m+n}(x)$, ... and $f^{-1}(x)$ must denote that quantity whose function f is x."

646. G. Peano[7] writes $y=f(x)$ and $x=\bar{f}(y)$, where $\bar{f}$ means the inverse function of f. This notation is free from the objection raised to that of Herschel and Bürmann (§533, 645), and to $gd^{-1}\phi$, used as the inverse Gudermannian by some writers, for instance, by J. M. Peirce in his *Mathematical Tables* (1881, p. 42) and D. A. Murray in his *Differential and Integral Calculus* (1908, p. 422), but by others more appropriately marked $\lambda(\phi)$.

B. SYMBOLS FOR SOME SPECIAL FUNCTIONS

647. *Symmetric functions.*—Attention was directed in § 558 to Leibniz' dot notation for symmetric functions. Many writers used no special symbol for symmetric functions; the elementary functions were placed equal to the respective coefficients of the given equation, due attention being paid to algebraic signs. This course was followed

[1] *Loc. cit.; Œuvres*, Vol. II (Paris, 1868), p. 40.

[2] *N. Mémoires d. l'acad. r. d. scienc. et bell.-lett. de Berlin*, année 1772; *Œuvres*, Vol. III (Paris, 1869), p. 442.

[3] *N. Mémoires d. l'acad. r. d. scienc. et bell.-lett. de Berlin*, année 1774; *Œuvres*, Vol. IV (Paris, 1869), p. 30. See also p. 591.

[4] Wolfgang Bolyai, *Az arithmetica eleje* (Maros-Vásárhelyt, 1830). See B. Boncompagni, *Bullettino*, Vol. I (1868), p. 287.

[5] A. M. Legendre, *Éléments de géométrie* (éd. par J. B. Balleroy ... avec notes ... par M. A. L. Marchand; Bruxelles, 1845), p. 188.

[6] J. F. W. Herschel, *Calculus of Finite Differences* (Cambridge, 1820), p. 5. Herschel says that he first explained his notation for inverse functions in the *Philosophical Transactions* of London in 1813 in his paper "On a Remarkable Application of Cotes's Theorem," but that he was anticipated in this notation by Burmann.

[7] G. Peano, *Lezioni di analisi infinitesimale*, Vol. I (Torino, 1893), p. 8.

by Cauchy[1] in 1821. Kramp[2] denoted these functions by $\int A$, where A is the general term.

Charles Babbage[3] wrote expressions like $F[\bar{x}, \overline{\psi(x)}]$, where the bars above the x and $\psi(x)$ were intended to specify that the function F is symmetric with respect to these two quantities. E. Meissel[4] represents by $(x|\varphi x)$ a symmetric function of the quantities x and $\varphi(x)$, where $\varphi(x)$ is determined by the equation $(x|\varphi x)=0$.

648. J. R. Young[5] in 1843 puts "for abridgment $\Sigma_m = a_1^m + a_2^m + a_3^m + \ldots\ldots a_n^m$," "where $a_1, a_2, \ldots\ldots a_n$ are the roots of an equation." J. A. Serret[6] said: "Let $a, b, c, \ldots\ldots, k, l$ be the m roots of an equation $X=0$ of the degree m, and let us consider a double symmetric function, of which one term is $a^\alpha b^\beta$; the function in question being determined when one term is known, we represent, for brevity, by $\Sigma a^\alpha b^\beta$, and we continue to designate by s_α the sum of the αth powers of all the roots." W. S. Burnside and A. W. Panton state: "It is usual to represent a symmetric function by the Greek letter Σ attached to one term of it, from which the entire expression may be written down."[7]

649. *Gamma and beta functions.*—Both of these functions are the creations of L. Euler, but their names and their symbolic representations originated in the nineteenth century. Euler mentions the gamma function in a letter of October 13, 1729, to Goldbach,[8] also in a paper of 1730[9] and in subsequent writings. Over sixty years later Legendre[10] gave the function its sign and corresponding name. In one place he says:[11] "... l'intégrale $\int dx\left(\log\frac{1}{x}\right)^{a-1}$, dans laquelle nous supposerons que a est positif et plus petit que l'unité. Cette quantité étant simplement fonction de a, nous désignerons par $\Gamma(a)$." It was

[1] A. L. Cauchy, *Cours d'analyse* (Paris, 1821), p. 70, 71.

[2] C. Kramp, *Élémens d'arithmétique universelle* (Cologne, 1808), "Notations."

[3] Gergonne, *Annales de mathématiques* (Nismes), Vol. XII (1819–20), p. 81; Charles Babbage, *Solutions of Functional Equations* [1820], p. 2.

[4] E. Meissel, *Crelle's Journal*, Vol. XLVIII (1854), p. 303.

[5] J. R. Young, *Algebraical Equations* (2d ed.; London, 1843), p. 416.

[6] J. A. Serrett, *Cours d'algèbre supérieure* (2d ed.; Paris, 1854), p. 11, 12.

[7] W. S. Burnside and A. W. Panton, *Theory of Equations* (4th ed.; Dublin and London), Vol. I (1899), p. 47.

[8] P. H. Fuss, *Correspondance math. et phys.*, Vol. I (1843), p. 3, 4.

[9] *Comment. acad. Petropolitanae ad annos 1730 et 1731*, Vol. V, p. 36–57.

[10] See our § 448. See also A. M. Legendre, *Exercices de calcul intégral;* Vol. I (Paris, 1811), p. 277.

[11] A. M. Legendre, *Traité des fonctions elliptiques*, Vol. II (Paris, 1826), p. 406.

in the same paper of 1730 that Euler gave what we now call the "beta function." He says:[1] "Sit proposita haec formula $\int x^e dx(1-x)^n$ vicem termini generalis subiens, quae integrata ita, ut fiat$=0$, si sit $x=0$, et tum posito $x=1$, dat terminum ordine n progressionis inde ortae." About a century after Euler's first introduction of this function, Binet wrote the integral in the form $\int_0^1 x^{p-1}dx(1-x)^{q-1}$ and introduced the Greek letter beta, B. Considering both beta and gamma functions, Binet said:[2] "Je désignerai la première de ces fonctions par $B(p, q)$; et pour la seconde j'adoperai la notation $\Gamma(p)$ proposée par M. Legendre." Legendre had represented the beta function by the sign $\left(\frac{p}{q}\right)$.

650. Gauss[3] expressed the gamma function by Πz and defined $\Pi(k, z)=\frac{k^z \Pi k \cdot \Pi z}{\Pi(k+z)}$; Γz means the same as $\Pi(z-1)$. Weierstrass called the reciprocal of $\Gamma(1+u)$ the *factorielle* of u and wrote it $Fc\, u$.[4]

651. *Elliptic functions.*—In his treatise on elliptic functions, Legendre[5] starts out by letting R represent the radical $\sqrt{(\alpha+\beta x+\gamma x^2+\delta x^3+\epsilon x^4)}$, Π^m the integral $\int \frac{x^m dx}{R}$, and Γ^k the integral $\int \frac{dx}{(1+nx)^k R}$. A little later[6] he denotes (taking the modulus $c<1$) the radical $\sqrt{(1-c^2 \sin^2 \varphi)}$ by Δ, also by $\Delta(\varphi)$ and by $\Delta(c, \varphi)$. He represents the general formula $\int \frac{A+B \sin^2 \varphi}{1+n \sin^2 \varphi} \cdot \frac{d\varphi}{\Delta}$ by H, and special cases of it, namely (p. 15), $\int \Delta d\varphi$, by E, $\int \frac{d\varphi}{\Delta}$ by F. He writes also $F(\varphi)$, $F(c, \varphi)$, $E(c, \varphi)$, $\Pi(n, c, \varphi)$.

652. Jacobi in 1829 introduced a new notation, placing[7] $\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}=u$, where φ is the amplitude of the function u.

[1] Quoted by M. Cantor, *op. cit.*, Vol. III (3d ed.), p. 653.

[2] Jacques P. M. Binet in *Jour. école polyt.*, Vol. XVI (1839), p. 131.

[3] K. F. Gauss, *Werke*, Vol. III (Göttingen, 1866), p. 146.

[4] *Crelle's Journal*, Vol. LI (1856), p. 7; Weierstrass, *Mathematische Werke*, Vol. I (1894), p. 161.

[5] A. M. Legendre, *Traité des fonctions elliptiques* (Paris), Vol. I (1825), p. 4, 5, 17.

[6] *Op. cit.*, p. 11, 14, 35, 107.

[7] C. G. J. Jacobi's *Fundamenta nova*, in *Gesammelte Werke*, Vol. I (Berlin, 1881), p. 81, 82.

Hence this angle is called "ampl u," or briefly "$\varphi = \text{am}\, u$." Also when

$$u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

$x = \sin \text{am}\, u$. Writing

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = K,$$

he calls $K-u$ the complement of the function u, and "coam" the amplitude of the complement, so that $\text{am}(K-u) = \text{coam}\, u$. In partial conformity with Legendre, Jacobi writes $\Delta\, \text{am}\, u = \frac{d\, \text{am}\, u}{du} = \sqrt{1-k^2 \sin^2 \text{am}\, u}$. He uses k in place of Legendre's c; Jacobi marked the complement of the modulus k by k' so that $kk+k'k'=1$. He uses the expressions "sin am u," "sin coam u," "cos am u," "cos coam u," "Δ am u," "Δ coam u," etc. The Greek letter Z was introduced by Jacobi thus:[1] "*E. Cl^i.* Legendre notatione erit, posito $\frac{2Kx}{\pi} = u$, $\varphi = \text{am}\, u$:

$$\mathrm{Z}(u) = \frac{F^I E(\varphi) - E^I F(\varphi)}{F^I}."$$

This is Jacobi's zeta function.

653. If in Jacobi's expression for K, k' is written for k, the resulting expression is marked K'. Weierstrass and Glaisher denoted $K-E$ by J. With Weierstrass $J' = E'$; with Glaisher[2] $J' = K' - E'$. Glaisher writes also $G = E - k'^2K$ and $G' = E' - k^2K'$.

Using Jacobi's zeta function, Glaisher[3] introduces three functions, ez x, iz x, gz x, defined thus: ez $x = \frac{E}{K}x + \mathrm{Z}(x)$, iz $x = -\frac{J}{K}x + \mathrm{Z}(x)$, gz $x = \frac{G}{K}x + \mathrm{Z}(x)$. Glaisher defines also Ae_sx, Ai_sx, Ag_sx; for example, $Ai_sx = e^{\int_0^x iz\, x\, dx}$. The Ai_sx is the same as Weierstrass' Al_sx.

654. To some, Jacobi's notation seemed rather lengthy. Accordingly, Gudermann,[4] taking $\varphi = \text{am}\, u$ and $u = \text{arg am}\, (\varphi)$, suggested the

[1] *Op. cit.*, Vol. I, p. 187.

[2] J. W. L. Glaisher in *Quarterly Journal of Mathematics*, Vol. XX (London, 1885), p. 313; see also Vol. XIX (1883), p. 145–57.

[3] J. W. L. Glaisher, *ibid.*, Vol. XX, p. 349, 351, 352.

[4] C. Gudermann in *Crelle's Journal*, Vol. XVIII (Berlin, 1838), p. 14.

more intense abbreviations[1] sn u for sin am u, cn u for cos am u, tn u for tang am u, dn u for $\sqrt{(1-k^2 \sin^2 \text{am } u)}$. The argument $K-u$, which is the complement of u, appears in the following abbreviations used by Guderman (p. 20): amc $u=$am $(K-u)$, snc $u=$sin amc $u=$ sn $(K-u)$, and similarly in cnc u, tnc u, dnc u. Before Gudermann, Abel[2] had marked sn u by the sign $\lambda(\theta)$. Gudermann's symbols sn u, cn u, dn u were adopted by Weierstrass in a manuscript of 1840, where he considered also functions which he then marked $A(u)$, $B(u)$, $C(u)$, $D(u)$, but which he and others designated later[3] by $\text{Al}(u)_1$, $\text{Al}(u)_2$, $\text{Al}(u)_3$, $\text{Al}(u)$ and called "Abelian functions." In 1854[4] Weierstrass called certain $2n+1$ expressions $\text{al}(u_1, u_2, \ldots .)_0$, $\text{al}(u_1, u_2, \ldots .)_1$, etc., Abelian functions; "it is they which correspond perfectly to the elliptic functions sin am u, cos am u, Δ am u." He shows the relation[5] $\text{al}(u_1, u_2, \ldots .)_a=\text{Al}(u_1, u_2, \ldots .)_a : \text{Al}(u_1, u_2, \ldots .)$. In 1856 he changed the notation slightly[6] from $\text{al}(u_1, u_2, \ldots .)_0$ to $\text{al}(u_1, \ldots .)_1$, etc.

If $\int^{\infty}\frac{dx}{2\sqrt{X}}=\int\frac{ds}{\sqrt{S}}=u$, where X is a cubic expression in x, $x=s+f$, and f is so chosen that $S=4s^3-g_2s-g_3$, then s is an elliptic function of u, of the second degree, denoted by $\wp u$ in the notation of Weierstrass.[7] He marked its derivative $\wp^1 u$. Connected with these is an analytic function which he[8] marked σu and employed in the expression of elliptic functions as quotients of integral transcendental functions. He defines it thus:

$$\sigma u=u\Pi'_w\left(1-\frac{u}{w}\right)e^{\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}}, \qquad w=2\mu\omega+2\mu'\omega',$$
$$\mu, \mu'=0, \pm 1, \pm 2, \pm 3, \ldots . \pm\infty,$$

[1] A. C. Dixon, in his *Elliptic Functions* (London, 1894), p. 70 n., reminds us that "the notation sg u, cg u, for sn $(u, 1)$, cn $(u, 1)$ is sometimes used, in honour of Gudermann."

[2] N. H. Abel, *Crelle's Journal*, Vol. IV (1829), p. 244.

[3] K. Weierstrass, *Mathematische Werke*, Vol. I, p. 5, 50; *Crelle's Journal*, Vol. LII (1856), p. 349.

[4] K. Weierstrass, *Crelle's Journal*, Vol. XLVII (1854), p. 291.

[5] K. Weierstrass, *op. cit.*, p. 300.

[6] K. Weierstrass, *Crelle's Journal*, Vol. LII (1856), p. 313.

[7] A. G. Greenhill, *Applications of Elliptic Functions* (London, 1892), p. 43; H. A. Schwarz, *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen des Herrn K. Weierstrass* (Göttingen, 1885), p. 2.

[8] R. Fricke in *Encyclopädie der Math. Wissenschaften*, Band II, 2 (Leipzig, 1913), p. 268.

where the accent (′) is to serve as a reminder that the value $w=0$ is to be omitted.[1] Weierstrass obtains $\wp u = -\frac{d^2}{du^2}\log \sigma u$. The definitions of three Weierstrass functions $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ are given by Schwarz[2] and Fricke.

655. J. W. L. Glaisher introduced a notation for the reciprocals of elliptic functions by simply interchanging the order of letters; he designated[3] the reciprocals of cn u, sn u, dn u by nc u, ns u, nd u, respectively. He marked the quotient cn u/dn u by cd u. He proceeded similarly with other quotients. According to this plan, sn u/cn u would be sc u, but, according to Greenhill,[4] it is more commonly denoted by tanam u, abbreviated to tn u, while cn u/sn u or cs u was denoted by cotam u or ctn u. Glaisher marks the other four quotients thus: sn u/dn u = sd u, cn u/dn u = cd u, dn u/sn u = ds u, dn u/cn u = dc u. This notation has not met with wide adoption.

Glaisher says:[5] "The function $Z(x)$ is especially related to the group sn x, cn x, dn x, and the functions $Z_1(x)$, $Z_2(x)$, $Z_3(x)$ are similarly related to the other three groups ns x, ds x, cs x; dc x, nc x, sc x; cd x, sd x, nd x, respectively. It thus appears that a two-letter notation for the four zeta functions, in which they are distinguished from each other by the four final letters n, s, c, d, would be more representative than one in which the distinction is made by means of the suffixes 0, 1, 2, 3. I therefore denote the four zeta functions by zn x, zs x, zc x, zd x, so that zn $x = Z(x)$, zs $x = Z_1(x)$, zc $x = Z_2(x)$, zd $x = Z_3(x)$."

656. *Theta functions.*—Jacobi, in his *Fundamenta nova functionum ellipticarum* (1829), introduced not only the $Z(u)$ function in course of the representation of integrals of the "2. Gattung" in the form of Fourier series, but also the new transcendent $\Theta(u)$ which appears in the treatment of integrals of the "3. Gattung." Dirichlet in his memorial address remarked: "Strangely such an important function has as yet no other name than that of transcendent Θ, which is the chance designation by which it first appears in Jacobi, and mathematicians would only fulfill an obligation of thankfulness if they were

[1] H. S. Schwarz, *op. cit.*, p. 5; K. Weierstrass, *Mathematische Werke*, Vol. II (Berlin, 1895), p. 245–55.

[2] H. A. Schwarz, *op. cit.*, p. 21; Fricke, *op. cit.*, p. 279.

[3] J. W. L. Glaisher in *Messenger of Mathematics*, Vol. XI (London and Cambridge, 1882), p. 86.

[4] A. G. Greenhill, *op. cit.*, p. 17.

[5] J. W. L. Glaisher in *Messenger of Mathematics*. Vol. XVIII (1889), p. 2.

to unite, and attach to it the name of Jacobi."[1] Jacobi says in his *Fundamenta nova* (1829):[2] "Designabimus in posterum per characterem $\Theta(u)$ expressionem:

$$\Theta(u)=\Theta(0)e^{\int_0^u Z(u)du},$$

designante $\Theta(0)$ constantem, quam adhuc indeterminatam relinquimus, dum commodam eius determinationem infra obtinebimus." Jacobi expresses the elliptic functions by the functions Θ and H. The latter is called the "eta function." He writes,[3] for instance,

$$\sin \operatorname{am}\frac{2Kx}{\pi}=\frac{1}{\sqrt{k}}\cdot\frac{\mathrm{H}\left(\frac{2Kx}{\pi}\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}.$$

Later Jacobi gave lectures (elaborated in 1838 by C. W. Borchardt) in which the theta and eta functions and series, which he had reached at the end of his *Fundamenta nova*, were made the starting-point of a new treatment of elliptic functions.[4] He uses now the following symbolism:

$$\vartheta\ (x)=1-2q\cos 2x+2q^4\cos 4x-\ldots\ldots,$$
$$\vartheta_1(x)=2\sqrt[4]{q}\sin x-2\sqrt[4]{q^9}\sin 3x+\ldots\ldots,$$
$$\vartheta_2(x)=2\sqrt[4]{q}\cos x+2\sqrt[4]{q^9}\cos 3x+\ldots\ldots,$$
$$\vartheta_3(x)=1+2q\cos 2x+2q^4\cos 4x+\ldots\ldots,$$

or, when necessary, the fuller designations $\vartheta(x, q)$, $\vartheta_1(x, q)$, $\vartheta_2(x, q)$, $\vartheta_3(x, q)$, where q lies between 0 and 1.

Unfortunately, the literature of theta functions shows marked diversity of notations.

657. Weierstrass[5] wrote ϑ_0 in place of ϑ, and defined $\vartheta_0(v)=1-2h\cos 2v\pi+2h^4\cos 4v\pi-\ldots\ldots$; he defined $\vartheta_1(v)=2h^{\frac{1}{4}}\sin v\pi-2h^{\frac{9}{4}}\sin 3v\pi+\ldots\ldots$, and similarly for $\vartheta_2(v)$, $\vartheta_3(v)$.

When it seemed serviceable to indicate the two periods, 2ϖ and $2\varpi'$, of the functions, Weierstrass[6] wrote

$$\vartheta_\rho\left(v\,\middle|\,\frac{\varpi'}{\varpi}\right)=\vartheta_\rho(v\,|\,\tau),\ \rho=1,\ 2,\ 3,\ 0\ ;$$

[1] Leo Königsberger, *Carl Gustav Jacob Jacobi* (Leipzig, 1904), p. 98.

[2] C. G. J. Jacobi, *Gesammelte Werke*, Vol. I (Berlin, 1881), p. 198, 257.

[3] *Op. cit.*, p. 224.

[4] C. G. J. Jacobi, *op. cit.*, Vol. I, p. 501.

[5] H. A. Schwarz, *op. cit.*, p. 41.

[6] H. A. Schwarz, *op. cit.*, p. 42.

he employed also

$$\Theta_\rho(u)=\Theta_\rho(u\,|\,\tilde{\omega},\,\omega')=2^{2\bar{\eta}\tilde{\omega}}\vartheta_\rho\left(v\,\middle|\,\frac{\tilde{\omega}'}{\omega}\right),\ u=2\tilde{\omega}v,\ \rho=1,\,2,\,3,\,0\,.$$

The notation of Weierstrass was followed by Halphen. Camille Jordan adopts the signs θ, θ_1, θ_2, θ_3 in the place of the Weierstrassian ϑ_1, ϑ_2, ϑ_0, ϑ_3, respectively. "Par le changement d'indices que nous avons pris la liberté de faire, on rétablit le parallélisme des notations entre ces fonctions et les σ correspondants."[1] Harkness and Morley, in their *Theory of Analytic Functions* (1898), page 242, represent Jordan's θ, θ_1, θ_2, θ_3 by ϑ, ϑ_1, ϑ_2, ϑ_3, respectively, so that Harkness and Morley's ϑ_3 is the same as the ϑ_3 of Weierstrass, and Harkness and Morley's ϑ, ϑ_1, ϑ_2 are equal to the Weierstrassian ϑ_1, ϑ_2, ϑ_0, respectively. Harkness and Morley remind[2] the reader that in their *Treatise on the Theory of Functions* they have a different notation from that in their *Analytic Functions*.

In 1858 Hermite[3] used a symbolism $\theta_{\mu,\nu}$ which, if $\omega=\tau$, yields the following equivalences with the signs of Weierstrass; $\theta_{0,0}(x)=\vartheta_3(x)$, $\theta_{0,1}(x)=\vartheta_0(x)$, $\theta_{1,0}(x)=\vartheta_2(x)$, $\theta_{1,1}(x)=i\vartheta_1(x)$.

G H. Cresse[4] prepared two tables exhibiting the various notations in theta functions followed by authors whom he cites in the sixth chapter of the third volume of Dickson's *History*. A most exhaustive statement of notations for theta functions is given by Brill and Noether.[5]

The sign $\theta(z)$ has also been used to represent what Poincaré[6] called the "thetafuchsian series and functions"; A. R. Forsyth[7] marks these functions by a capital letter $\Theta(z)$.

658. In the development of general theta functions involving p variables (arguments) the notation $\vartheta(v_1\,|\,v_2\,\ldots\,.\,|v_p)$ has been used; when the mark or characteristic $[{}^g_h]$ is introduced, and the arguments differ from each other only in their subscripts, the subscripts are

[1] Camille Jordan, *Cours d'analyse*, Vol. II (Paris, 1894), p. 411.

[2] Harkness and Morley, *Analytic Functions* (1898), p. 242.

[3] C. Hermite in *Journal de mathématiques* (Liouville; 2d ser.), Vol. III (1858), p. 26; H. A. Schwarz, *op. cit.*, p. 42.

[4] L. E. Dickson, *History of the Theory of Numbers*, Vol. III (Washington, 1923), chap. vi (written by G. H. Cresse), p. 93, 94.

[5] A. Brill and M. Noether, "Entwicklung d. Theorie d. alg. Functionen," *Jahresbericht d. d. Mathematiker Vereinigung*, Vol. III (Berlin, 1894), p. 500–520.

[6] H. Poincaré in *Acta mathematica*, Vol. I (Stockholm, 1882), p. 207, 208, 210.

[7] A. R. Forsyth, *Theory of Functions of a Complex Variable* (Cambridge, 1893), p. 642.

omitted and double parentheses are introduced so that the functions are represented[1] by the symbolism $\vartheta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]((v))$, or in case of a theta function of the nth order, by the symbolism $\Theta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]((v))$. Harkness and Morley[2] mark the p-tuple theta functions by $\theta(v_1, v_2, \ldots, v_p)$.

659. *Zeta functions.*—Reference has been made to Jacobi's zeta function, marked by the capital Greek letter Z (§ 653). The small letter zeta has been used to stand for the integral $\zeta(u) = -\int \wp(u)du$ by various writers.[3]

660. This same small letter ζ was used by B. Riemann[4] as early as 1857 to represent a function on which he based his analysis of prime numbers and which he introduced thus: "In this research there serves as point of departure the remark of Euler that the product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

if there are taken for p all prime numbers, for n all integral numbers. The function of the complex variable s which is represented by these two expressions, as long as they converge, I designate by $\zeta(s)$." This notation has maintained its place in number theory.

661. *Power series.*—Weierstrass[5] denotes by a German capital letter $\mathfrak{P}$ a power series; he marks a "Potenzreihe von x" by $\mathfrak{P}(x)$ and by $\mathfrak{P}(x-a)$, when a is the center of the circle of convergence. More commonly the Latin capital P is used. Harkness and Morley[6] remark: "The usual notation for such a series is $P(z)$."

662. *Laplace, Lamé, and Bessel functions.*—What are called Laplace's coefficients and functions was first worked out by Legendre and then more fully by Laplace.[7] If $(1-2ax+x^2)^{-\frac{1}{2}}$ is expanded

[1] A. Krazer and W. Wirtinger in *Encyklopädie der Math. Wissensch.*, Vol. II (1921), p. 637, 640, 641.

[2] Harkness and Morley, *Theory of Functions* (1893), p. 459. On p. 460 and 461 they refer to differences in notation between Riemann, and Clebsch and Gordan, with regard to the lower limits for the integrals there involved.

[3] A. R. Forsyth, *Theory of Functions of a Complex Variable* (Cambridge, 1893), p. 250, 251; R. Fricke in *Encyklopädie d. math. Wissensch.*, Vol. II, 2 (Leipzig, 1913), p. 258; R. Fricke, *Elliptische Funktionen* (1916), p. 168; H. Burkhardt, *Funktionen-theoretische Vorlesungen*, Vol. II (Leipzig, 1906), p. 49.

[4] B. Riemann, *Gesammelte Werke* (Leipzig, 1876), p. 136.

[5] K. Weierstrass, *Mathem. Werke* (Berlin), Vol. II (1895), p. 77.

[6] J. Harkness and Morley, *Theory of Functions* (New York, 1893), p. 86.

[7] *Mémoires de math. et phys., présentés à l'académie r. d. sciences par divers savans*, Vol. X (Paris, 1785).

in a series of ascending powers of α, the coefficient of α^n is a function of x, often called "Legendre's coefficient." Dirichlet[1] represents it by P_n, E. Heine[2] by $P^{(n)}$, or, when no confusion with exponents is possible, simply by P^n, while Todhunter[3] denotes it by P_n or $P_n(x)$, and says, "French writers very commonly use X_n for the same thing." When for x there is substituted the value $\cos\gamma = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1 \cdot \cos(\phi - \phi_1)$, where both θ and ϕ are regarded as variables, while θ_1 and ϕ_1 are constants, then the coefficient is called "Laplace's coefficient" and is marked Y_n by Todhunter.[4] Laplace's coefficients are particular cases of Laplace's functions which satisfy a certain partial-differential equation, and are marked by Todhunter[5] X_n or Z_n, n denoting the order of the function. Heine[6] marks them P^n $(\cos\gamma)$.

663. Analogous to Laplace's functions are those of Lamé which Heine[7] marks $E_s(\mu)$. Lamé's functions must satisfy a linear-differential equation of the second order.

664. Bessel[8] defined the function now known by his name by the following definite integral:

$$\text{“}\int \text{Cos}\,(h\epsilon - k\,\text{Sin}\,\epsilon)\,d\epsilon = 2\pi I_k^h,\text{”}$$

where h is an integer and the limits of integration are 0 and 2π. His I_k^h is the same as the modern $J_h(k)$, or rather $J_n(x)$. O. Schlömilch,[9] following P. A. Hansen,[10] explained the notation $J_{\lambda,\,\pm n}$ where λ signifies the argument and $\pm$ the index of the function. Schlömilch usually omits the argument. Watson[11] points out that Hansen and Schlömilch express by $J_{\lambda,\,n}$ what now is expressed by $J_n(2\lambda)$. Schläfli[12] marked it $\overset{n}{J}(x)$. Todhunter[13] uses the sign $J_n(x)$. $J_n(x)$ is known as the "Bessel

[1] L. Dirichlet in *Crelle's Journal*, Vol. XVII (1837), p. 35, 39.

[2] E. Heine, *Kugelfunctionen* (2d ed.; Berlin, 1878), p. 3.

[3] I. Todhunter, *Laplace's Functions, Lamé's Functions, and Bessel's Functions* (London, 1875), p. 3.

[4] I. Todhunter, *op. cit.*, p. 131.

[5] I. Todhunter, *op. cit.*, p. 152.

[6] E. Heine, *op. cit.*, p. 302.

[7] E. Heine, *op. cit.*, p. 358.

[8] F. W. Bessel in *Abhandlungen Akademie der Wissensch. (Math. Kl.) 1824* (Berlin, 1826), p. 22, 41.

[9] O. Schlömilch, *Zeitschr. d. Math. u. Phys.*, Vol. II (1857), p. 137.

[10] P. A. Hansen, *Ermittelung der absoluten Störungen*, 1. Theil (Gotha, 1843).

[11] G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922), p. 14.

[12] L. Schläfli, *Mathemat. Annalen*, Vol. III (1871), p. 135

[13] I. Todhunter, *op. cit.*, p. 285.

function of the first kind of order n," while $Y^n(x)$, an allied function introduced in 1867 by Karl Neumann,[1] is sometimes called "Newmann's Bessel function of the second kind of order n." It[2] is sometimes marked $Y_n(x)$.

Watson says: "Functions of the types $J\pm(n+\frac{1}{2})^{(z)}$ occur with such frequency in various branches of Mathematical Physics that various writers have found it desirable to denote them by a special functional symbol. Unfortunately no common notation has been agreed upon and none of the many existing notations can be said to predominate over the others." He proceeds to give a summary of the various notations.[3]

In his *Theory of Bessel Functions*, pages 789, 790, Watson gives a list of 183 symbols used by him as special signs pertaining to that subject.

665. *Logarithm integral, cosine integral, etc.*—Soldner[4] uses the symbol li. a, from the initial letters of *logarithme integral*, to stand for $\int_0^a dx{:}\log x$. This notation has been retained by Bretschneider,[5] De Morgan,[6] and others.

As regards the sine integral and cosine integral, first considered by Mascheroni, Bessel[7] in 1812 still writes the expressions $\int \frac{dx}{x}\sin x$ and $\int \frac{dx}{x}\cos x$ without special signs to represent them. Bretschneider[8] marked the cosine integral by "ci x," and the sine-integral by "si x." He denoted the hyperbolic functions $\int \text{Cos}\, x \cdot \frac{dx}{x}$ by "$\mathfrak{C}\mathfrak{J}$" and $\int \text{Sin}\, x \cdot \frac{dx}{x}$ by "$\mathfrak{S}\mathfrak{J}$." Later still Schlömilch[9] denotes $\int_0^1 \frac{\sin w\theta}{\theta}\, d\theta$

[1] Karl Neumann, *Theorie der Bessel'schen Funktionen* (Leipzig, 1867), p. 41, 52; A. Gray and G. B. Mathews, *Treatise on Bessel Functions* (2d ed.; London, 1922), p. 15; G. N. Watson, *op. cit.*, p. 67.

[2] Gray and Mathews, *op. cit.* (1922), p. 15.

[3] G. N. Watson, *op. cit.* (1922), p. 55, 56.

[4] J. Soldner, *Théorie et tables d'une nouvelle fonction transcendante* (Munich, 1809); see also F. W. Bessel, *Abhandlungen*, Vol. II (Leipzig, 1876), p. 331.

[5] C. A. Bretschneider in *Crelle's Journal*, Vol. XVII (Berlin, 1837), p. 257.

[6] A. de Morgan, *Differential and Integral Calculus* (London, 1842), p. 660.

[7] F. W. Bessel, *Abhandlungen*, Vol. II (Leipzig, 1876), p. 341.

[8] C. A. Bretschneider, *Grunert's Archiv*, Vol. III (1843), p. 30.

[9] O. Schlömilch in *Crelle's Journal*, Vol. XXXIII (1846), p. 318.

by Si(w), and writes similarly Ci(w) and the exponential integral Ei(w), where Ei(w) = li(e^w). This notation was used by J. W. L. Glaisher[1] in 1870, except that he drops the parentheses and writes simply Si x, etc.

666. Niels Nielsen in 1904[2] marked the sine integral and cosine integral by "$\mathrm{S_i}(x)$" and "$\mathrm{C_i}(x)$," and in 1906[3] by "ci(x)" and "si(x)." Closely allied with these transcendents are the integrals of Kramp,[4] $\int_0^x e^{-x^2}dx$, which Nielsen in 1904 represents by K(x) and in 1906 by L(x), and the integrals of Fresnel,[5] $\int dz \cos (a-z)^2$ and $\int dz \sin (az^2)$, for which he used no sign and which are variously represented in later publications. Thus Nielsen[6] writes $F_1(x)$ and $F_2(x)$; Gray and Mathews[7] use C and S as does also Nielsen[8] in 1906.

In the Alphabetical Index of Nielsen's *Cylinderfunktionen* (1904), under the word "Funktion," the reader will find a long list of signs of functions which Nielsen used in that work.

SYMBOLS IN MATHEMATICAL LOGIC

667. *Some early symbols.*—Leibniz is generally regarded as the earliest outstanding worker in the field of symbolic logic. But before him certain logical symbols were introduced by Hérigone, in his *Cursus mathematicus* (Paris, 1634, 1644), and by Rahn, in his *Teutsche Algebra* (1659). Hérigone says in his Preface: "I have invented a new method of giving demonstrations, brief and intelligible, without the use of any language. . . . . The demonstration is carried on from beginning to end by a continuous series of legitimate consequences, necessary and immediate, each contained in a short line, and they can be readily resolved into syllogisms. . . . ." Hérigone employs "hyp." for "from the hypothesis it follows"; "constr." for "from the construction one has"; "concl." for "conclusion"; "arbitr." for "taking arbitrarily"; "20.1" for "from proposition 20 of the first book it follows." Most

[1] *Philosophical Transactions* (London, 1870), p. 367.

[2] N. Nielsen, *Handbuch der Cylinderfunktionen* (Leipzig, 1904), p. 71, 72.

[3] N. Nielsen, *Theorie des Integrallogarithmus* (Leipzig, 1906), p. 6.

[4] C. Kramp in *Mémoires d. l'acad. d. sciences*, Vol. V (Paris, 1818), p. 434. Reference taken from N. Nielsen, *Cylinderfunktionen* (1904), p. 72.

[5] A. J. Fresnel, *Analyse des réfractions astronomiques et terrestres* (1798). Also *Œuvres*, Vol. I (Paris, 1866), p. 176, 177.

[6] N. Nielsen, *Cylinderfunktionen* (1904), p. 72.

[7] A. Gray and G. B. Mathews, *op. cit.* (1922), p. 219.

[8] N. Nielsen, *Integrallogarithmus* (1906), p. 8.

of these abbreviations are hardly new with Hérigone; they were used more or less before him, in editions of Euclid. Included in his scheme of symbolic writing are his ideographic symbols for algebra and geometry which we noted in § 189.[1]

The symbols of Rahn have been listed in § 194. Logical, as distinct from operational, are his symbols for "therefore" and for "impossible." The wide use of the sign for "therefore" in English and American books impels us to trace its history more fully.

668. *The sign for "therefore."*—In Rahn's *Teutsche Algebra* (Zurich, 1659) one finds both ∴ and ∵ for "therefore," the ∴ predominating. In the English translation of 1668 both signs are found, but ∵ predominates. On page 37 of the translation one reads: "Which Deduction is hereafter noted with [∵] that is as much as [*ergo*.] only [∵] is set in the work, [*ergo*] in the Margin." It should be noted that with Rahn and with many writers of the eighteenth century the three dots were used especially in connection with the process of finding the products of means and extremes of a proportion. Thus, Thomas Walter[2] says: "∵ Therefore; signifying the product of the two extremes is equal to that of the means." William Jones[3] writes the sign :· and uses it in operating with proportions and also equations.

The ˙˙ as signifying "therefore" occurs in Oughtred's *Opuscula mathematica* (1677) (§ 181), in texts by Alexander[4] and Ward,[5] Ronayne,[6] and Birks.[7] Both forms ∵ and ∴ are found in Jones, Woodhouse,[8] and Simpson.[9] Stone[10] gives the form :· . During the eighteenth century ∵ was used for "therefore" about as often as was ∴ .

669. *The sign for "because."*—We have not been able to find the use of ∵ for "because" in the eighteenth century. This usage seems to have been introduced in Great Britain and the United States in the nineteenth century. It is found in the *Gentleman's Mathematical*

[1] See also Gino Loria's article, "La logique mathématique avant Leibniz," *Bull. d. scienc. math.* (G. Darboux; 2d ser.), Vol. XVIII (Paris, 1894), p. 107–12.

[2] Thomas Walter, *A New Mathematical Dictionary* (London, [n.d.]), p. 9.

[3] William Jones, Synopsis (London, 1706), p. 143, 145.

[4] John Alexander, *Synopsis algebraica* (London, 1693), p. 7.

[5] John Ward, *Young Mathematician's Guide* (1707), p. 189, 371, etc.

[6] Philip Ronayne, *Treatise of Algebra* (London, 1727), p. 3.

[7] John Birks, *Arithmetical Collections* (London, 1766), p. 482, etc.

[8] Robert Woodhouse, *Principles of Analytical Calculation* (Cambridge, 1803).

[9] Thomas Simpson, *Treatise of Fluxions* (London, 1737), p. 95.

[10] E. Stone, *New Mathematical Dictionary* (London, 1726, 1743), art. "Series."

Companion (1805).[1] It did not meet with as wide acceptance in Great Britain and America as did the sign ∴ for "therefore."

The signs ∴ for "therefore" and ∵ for "because" are both found in the *Elements* of Euclid edited by members of the University of Cambridge (1827), in Wright,[2] Nixon,[3] C. Smith,[4] Buddon,[5] Cockshott and Walters,[6] White,[7] Shutts,[8] Young and Schwartz,[9] Auerbach and Walsh,[10] and Macnie.[11] In mathematical publications on the European Continent the two symbols are of very rare occurrence, except in papers on symbolic logic.[12] The sign ∵ for "therefore" is used by T. Svedberg[13] in 1907.

670. *The program of Leibniz.*—The contribution of Leibniz to symbolic logic was a program, rather than actual accomplishment. He set up an ideal for others to approach. He came to regard logic "like a universal mathematics."[14] He advocated "a universal language" or "universal characteristic," or a "universal calculus." "The true method should furnish us with an Ariadne's thread, that is to say, with a certain sensible and palpable medium, which will guide the mind as do the lines drawn in geometry and the formulas for operations, which are laid down for the learner in arithmetic."[15] We quoted an extremely optimistic passage from Leibniz in § 530. His earliest publication, the *De arte combinatoria* (1666), printed when

[1] See *Mathematical Gazette*, Vol. XI (London, 1923), p. 275.

[2] J. M. F. Wright, *Euclid* (Cambridge, 1829).

[3] R. C. J. Nixon, *Geometry of Space* (Oxford, 1888), p. 25.

[4] C. Smith, *Elements of Algebra* (ed. Irving Stringham; New York, 1895), p. 18.

[5] E. Budden, *Elementary Pure Geometry* (London, 1904), p. 22.

[6] A. Cockshott and F. B. Walters, *Treatise on Geometrical Conics* (London, 1881).

[7] E. E. White, *Elements of Geometry* (New York, [1895]).

[8] George C. Shutts, *Plane and Solid Geometry* [1912], p. 13.

[9] John W. Young and Albert J. Schwartz, *Plane Geometry* (New York, [1915]).

[10] Matilda Auerbach and Charles B. Walsh, *Plane Geometry* (Philadelphia, [1920]), p. xi.

[11] John Macnie, *Elements of Geometry* (ed. E. E. White; New York, 1895), p. 10.

[12] Both ∴ ("therefore") and ∵ ("because") are found in Platon Poretsky's *Théorie des non-égalités* (Kazan, 1904), p. 22, 23.

[13] *Nova acta regiae societatis scientiarum Upsaliensis* (4th ser.), Vol. II, No. 1, p. 114.

[14] G. W. Leibniz, *Initia et Specimina scientiae novae generalis*, in C. I. Gerhardt's *Philosophische Schriften von Leibniz*, Vol. VII (Berlin, 1890), p. 65, also p. 59.

[15] Letters of 1677 or 1678 to Galloys, C. I. Gerhardt's *Philosophische Schriften von Leibniz*, Vol. VII (Berlin, 1890), p. 22.

he was only twenty years of age, is his first attempt in this direction. Leibniz himself regarded it later as of little value. The task which he outlined seemed really beyond the powers of one man. Some manuscripts of his contain later developments. The symbols he used are listed in § 538–64. Above we quoted from his *Initia et specimina scientiae novae generalis*. Recently, additional Leibnizian manuscripts on this topic have been found in the Hanover Library.[1] In a manuscript dated April, 1679,[2] Leibniz experiments on the use of number and on mathematical operations in matters of logic. He proposes that primitive concepts be symbolized by prime number and the combinations of two concepts by their product. "For example, if it be assumed that the term 'animal' be expressed by some number 2 (or more generally by a) and the term 'rational' by 3 (or more generally by r), then the term 'man' is expressed by the number 2.3, that is 6, or by the product of the multiplication of 2 and 3 (or more generally by the number ar)." The proposition—all S is P—will be true if the number which contains the concept S is exactly divisible by that which represents the concept P. Accordingly, a universal affirmative proposition may be symbolized by $\frac{S}{P}$ yielding an integral quotient without remainder. For a particular affirmative proposition it suffices that either $\frac{S}{P}$ or $\frac{P}{S}$ be an integer. For a universal negative it suffices that neither $\frac{S}{P}$ nor $\frac{P}{S}$ be an integer. Certain difficulties of this scheme led Leibniz to experiment on the use of negative numbers, but he finally abandoned the plan of representing concepts by numbers.

Later fragments are included in plans for an encyclopedia[3] on science to be developed in terms of a universal characteristic, dealing with the equivalence of concepts. C. I. Lewis says:[4] "In these fragments, the relations of equivalence, inclusion, and qualification of one concept by another, or combination, are defined and used." As

[1] These fragments are contained in Louis Couturat, *Opuscules et fragments inédits de Leibniz; extraits des manuscrits de la Bibliothèque royale de Hanovre* (Paris, 1903). Two fragments from Leibniz are translated into English in C. I. Lewis' *Survey of Symbolic Logic* (Berkeley: University of California Press, 1918), p. 373–87.

[2] L. Couturat, *Opuscules, etc.* (1903), p. 42, 43.

[3] Gerhardt, *Philosoph. Schriften von Leibniz*, Vol. VII (1890), XVI–XX, p. 208–47.

[4] C. I. Lewis, *op. cit.*, p. 13.

the expositions of Leibniz deal with questions of logical theory without the use of new symbols, we must pass them by.

In another place[1] Leibniz formulates the symbolism for the four propositions (using the Cartesian ∞ for equality):

1. All A is B, that is $AB \infty A$;
2. Some A is not B, that is AB not ∞ A;
3. No A is B, that is AB does not exist, or AB not ∞ AB *Ens;*
4. Some A is B, that is AB exists, or $AB \infty AB$ *Ens.*

In (3) and (4) the AB represents the *possible* AB's or the AB "in the region of ideas," while the AB *Ens* represents the existing AB's, or actual members of the class AB. Here and in other fragments of Leibniz no new symbols are used, but different interpretations are given to old symbols.[2]

The breadth of view which characterized these plans precluded the possibility of their complete execution by Leibniz without the help of others. He invited the co-operation of others, but without effective response. His inability to secure promising pupils is voiced in a letter[3] of 1714: "I have spoken of my Spécieuse Générale to Mr. le Marquis de l'Hospital and to others, but they have given it no more attention than if I had related a dream. It seemed that I must back it up by some palpable example, for this purpose one would have to fabricate at least a part of my characteristic, which is not easy, especially at my age and without conversing with persons who could stimulate and assist me in researches of this sort."

A motive of disappointment at his failure to carry his plans to completion appears in another letter, written two years before his death:[4] "If I had been less distracted, or if I were younger, or assisted by well disposed young men, I could hope to give a sample of .... this language or universal writing, infinitely different from all hitherto projected, because the very characters .... would guide the reason; and errors, except those of fact, would be only errors of computation. It is very difficult to form or invent such a language or characteristic, but very easy to learn without a dictionary."

671. *The signs of J. H. Lambert.*—After Leibniz various attempts were made in the eighteenth century to develop a calculus of logic

[1] Gerhardt, *Philosoph. Schriften von Leibniz,* Vol. VII (1890), p. 214, 215. See also Lewis, *op. cit.*, p. 15.

[2] Confer C. I. Lewis, *op. cit.*, p. 5–18.

[3] *Leibnitii Opera philosophica* (ed. J. E. Erdmann; Berlin, 1840), p. 703.

[4] Letter in *Leibnitii Opera philosophica* (Erdmann, 1840), p. 701.

by J. A. Segner, Jakob Bernoulli, Gottfried Ploucquet, I. H. Tönnies, J. H. Lambert, G. J. von Holland, G. F. Castillon, and others. Peano[1] mentions also Ludovico Richeri[2] who in 1761 introduced a symbolism, for instance, $\cup$ and $\cap$, for "all" and "none." We give the symbols employed by J. H. Lambert:[3]

Gleichgültigkeit, equality	$=$	Particularity	$<$
Zusetzung, addition	$+$	Copula	$\sim$
Absonderung, abstraction	$-$	*Begriffe*, given concepts	a, b, c, d, etc.
Des Gegentheils, opposition	$\times$	Undeterminal concepts	n, m, l, etc.
Universality	$>$	Unknowns	x, y, z
		The genus	γ
		The difference	δ

Lambert deals with concepts, not classes. He uses the sign $a|b$, sometimes also $a:b$, to indicate that part of concept a which is different from concept b. In syllogisms the proposition "all A is B" has two cases: (1) $A=B$ in which the converse is $B>A$; (2) $A>B$, in which the converse is $B<A$. The proposition "some A is B" also has two cases: (1) $A<B$, in which the converse is $B>A$; (2) $A<B$, in which the converse is $B<A$. Lambert translates "some A" by mA. Hence $A>B$ is equivalent to $A=mB$; $A<B$ is equivalent to $nA=B$; $mA>B$ and $A<nB$ are together equivalent to $mA=kB$ and $lA=nB$, or equivalent to $pA=qB$. For "no A is B" Lambert writes $\frac{A}{m}=\frac{B}{n}$; by $\frac{A}{m}$ he means that the peculiar properties of the subject be taken away; by $\frac{B}{n}$ he means that the peculiar properties of the predicate be taken away. Lambert transforms fractions, though he carefully refrains from doing so in cases like $\frac{A}{m}=\frac{B}{n}$, which are supposed to represent universal negatives. Lambert's treatment of the negative is a failure.[4] Lambert represents non-logical or "metaphysical" relations

[1] G. Peano in *Rivista di matematica*, Vol. IV (1894), p. 120.

[2] L. Richeri in *Miscellanea Taurinensia*, Vol. II (1761), Part 3, p. 46–63.

[3] Johann Heinrich Lambert, "Sechs Versuche einer Zeichenkunst in der Vernunftlehre," *Logische und philosophische Abhandlungen* (ed. Joh. Bernoulli), Vol. I (Berlin, 1782). For a fuller account of Lambert see C. I. Lewis, *op. cit.*, p. 19–29.

[4] C. I. Lewis, *op. cit.*, p. 26.

by Greek letters. If f=fire, h=heat, α=cause, then he writes $f=\alpha::h$, where $::$ represents a relation which behaves like multiplication. From this equality he derives $\frac{f}{h}=\frac{\alpha}{\cdot}$, i.e., fire is to heat as cause is to effect. The dot represents here *Wirkung* ("effect").[1]

672. *Signs of Holland.*—G. J. von Holland in a letter[2] to Lambert points out a weakness in Lambert's algorithm by the following example: (1) All triangles are figures, $T=tF$; (2) all quadrangles are figures, $Q=qF$; (3) whence $F=\frac{T}{t}=\frac{Q}{q}$ or $qT=tQ$, which is absurd. In his own calculus Holland lets S represent the subject, P the predicate, p, π undetermined positive variable numbers. He interprets $\frac{S}{p}=\frac{P}{\pi}$ as meaning a part of S is a part of P. If $p=1$, then S/p is as many as all S. The sign $\frac{1}{\infty}$ is the same as 0; $\frac{S}{1}=\frac{P}{\infty}$ means "all S is not P." Holland's calculus does not permit his equations to be cleared of fractions.

673. *Signs of Castillon.*—The next writer on a calculus of logic was G. F. Castillon[3] who proposed a new algorithm. He lets S, A, etc., represent concepts in intension, M an indeterminate, $S+M$ the synthesis of S and M, $S-M$ the withdrawal of M from S, so that $S-M$ is a genus concept in which S is subsumed, $S+M$ a species concept. Accordingly, "all S is A" is represented by $S=A+M$; "no S is A" is represented by $S=-A+M=(-A)+M$, that is, withdrawing A from M is "no A." The real particular affirmative "some S is A" is expressed by $A=S-M$ and is the converse of $S=A+M$; the illusory particular affirmative is denoted by $S=A\mp M$ and means that in "some S is A" some S is got from A by the abstraction $S=A-M$ when in reality it is A that is drawn from S by the abstraction $S=A+M$. Hence this judgment puts $-M$ where there should be $+M$; this error is marked by $S=A\mp M$. Castillon's notation works out fairly well, but its defect lies in the circumstance that the sign $-$ is used both for abstraction and for negation.

674. *Signs of Gergonne.*—A theory of the meccanisme du raisonnement was offered by J. D. Gergonne in an *Essai de dialectique ration-*

[1] Our account of Lambert is taken from C. I. Lewis, *op. cit.*, p. 18–29.

[2] *Johann Lambert's deutscher Gelehrten Briefwechsel* (herausgegeben von J. Bernoulli), Brief III, p. 16 ff.; C. I. Lewis, *op. cit.*, p. 29, 30.

[3] G. F. Castillon, "Sur un nouvel algorithme logique," *Mémoires de l'académie des sciences de Berlin, 1803*, *Classe de philos. speculat.*, p. 1–14.

nelle;[1] there the symbol H stands for complete logical disjunction, X for logical product, I for "identity," C for "contains," and $\supset$ for "is contained in."

675. *Signs of Bolyai.*—Wolfgang Bolyai used logical symbols in his mathematical treatise, the *Tentamen*,[2] which were new in mathematical works of that time. Thus $A \doteq B$ signified A absolutely equal to B; $A \overset{\cdot}{\underline{=}} B$ signified A equal to B with respect to content; $A(=B$ or $B=)A$ signified that each value of A is equal to some value of B; $A(=)B$ signified that each value of A is equal to some value of B, and vice versa.

676. *Signs of Bentham.*—The earlier studies of logic in England hardly belong to symbolic logic. George Bentham[3] in 1827 introduces a few symbols. He lets = stand for identity, $||$ for diversity, t for *in toto* (i.e., universality), p for "partiality." Accordingly, "$tX=tY$" means X *in toto* $=$ Y *in toto*; "$tX=pY$" means X *in toto* $=$ Y *ex parte*;

$$\text{“}tX \,||\, {}^{t}_{p}Y\text{” means } X \textit{ in toto} \,||\, Y \begin{cases} \textit{in toto} \\ \text{or} \\ \textit{ex parte} \end{cases} .$$

677. *Signs of A. de Morgan.*—The earliest important research in symbolic logic in Great Britain is that of *De Morgan*.[4] He had not seen the publications of Lambert, nor the paper of J. D. Gergonne,[5] when his paper of 1846 was published. In 1831 De Morgan[6] used squares, circles, and triangles to represent terms. He does this also in his *Formal Logic*[7] but in 1831 he did not know[8] that Euler[9] had

[1] J. D. Gergonne in *Annales de mathématiques pures et appliquées*, Vol. VII (Nismes, 1816–17), p. 189–228. Our information is drawn from G. Vacca's article in *Revue de mathématiques*, Vol. VI (Turin, 1896–99), p. 184.

[2] Wolfgangi Bolyai de Bolya, *Tentamen .... in elementa matheseos* (2d ed.), Vol. I (Budapest, 1897), p. xi.

[3] George Bentham, *Outline of a New System of Logic* (London, 1827), p. 133.

[4] Augustus de Morgan, *Formal Logic* (London, 1847); five papers in the *Transactions of the Cambridge Philosophical Society*, Vol. VIII (1846), p. 379–408; Vol. IX (1850), p. 79–127; Vol. X (1858), p. 173–230; Vol. X (1860), p. 331–58; Vol. X (1863), p. 428–87.

[5] J. D. Gergonne, "Essai de dialectique rationelle," *Annales de mathématique*. (Nismes, 1816, 1817).

[6] De Morgan, "The Study and Difficulties of Mathematics," *Library of Useful Knowledge* (1831), p. 71–73.

[7] De Morgan, *Formal Logic* (1847), p. 8, 9.

[8] *Ibid.*, p. 323, 324.

[9] L. Euler, *Lettres à une Princesse d'Allemagne sur quelques sujets de Physique et de Philosophie* (Petersburg, 1768–72), Lettre CV.

used circles in the same manner. In his *Formal Logic*, De Morgan uses $X)$ or $(X$ to indicate distribution, and $X($ or $)X$ no distribution. Expressing in his own language (p. 60): "Let the following abbreviations be employed:

$X)Y$ means 'every X is Y'; $X.Y$ means 'no X is Y'

$X:Y$ means 'some Xs are not Ys'; XY means 'some Xs are Ys'."

He lets (p. 60) A stand for the universal affirmative, I for the particular affirmative, E for the universal negative, and O for the particular negative. He lets x, y, z, be the negatives or "contrary names" of the terms X, Y, Z. The four forms A_1, E_1, I_1, O_1 are used when choice is made out of X, Y, Z; the forms A', E', I', O' are used whence choice is made out of x, y, z. On page 61 he writes identities, one of which is "$A_1X)Y = X.y = y)x$," that is, every A_1X is $Y =$ no X is y = every y is x. On page 115: "P,Q,R, being certain names, if we wish to give a name to everything which is all three, we may join them thus, PQR: if we wish to give a name to everything which is either of the three (one or more of them) we may write P,Q,R: if we want to signify anything that is either both P and Q, or R, we have PQ,R. The contrary of PQR is p,q,r; that of P,Q,R is pqr; that of PQ,R is $(p,q)r$." In his *Syllabus*,[1] De Morgan uses some other symbols as follows:

"1. $X)\circ)Y$ or both $X))Y$ and $X)\cdot)Y$.
All Xs and some things besides are Ys.
2. $X||Y$ or both $X))Y$ and $X((Y$.
All Xs are Ys, and all Ys are Xs. . . .
5. $X|\cdot|Y$ or both $X)\cdot(Y$ and $X(\cdot)Y$.
Nothing both X and Y and everything one or the other."

De Morgan also takes L^{-1} as the converse of L.

The following quotation from De Morgan is interesting: "I end with a word on the new symbols which I have employed. Most writers on logic strongly object to all symbols, except the venerable *Barbara*, *Celarent*, etc., ... I should advise the reader not to make up his mind on this point until he has well weighed two facts which nobody disputes, both separately and in connexion. First, logic is the only science which has made no progress since the revival of letters; secondly, logic is the only science which has produced no growth of symbols."[2]

[1] A. de Morgan, *Syllabus of a Proposed System of Logic* (London, 1860), p. 22; C. I. Lewis, *op. cit.*, p. 41, 42.

[2] A. de Morgan, *op. cit.*, p. 72; see *Monist*, Vol. XXV (1915), p. 636.

678. *Signs of G. Boole.*—On the same day of the year 1847 on which De Morgan's *Formal Logic* was published appeared Boole's *Mathematical Analysis of Logic.*[1] But the more authoritative statement of Boole's system appeared later in his *Laws of Thought* (1854). He introduced mathematical operations into logic in a manner more general and systematic than any of his predecessors. Boole uses the symbols of ordinary algebra, but gives them different significations. In his *Laws of Thought* he employs signs as follows:[2]

"1st. Literal symbols, as x, y, etc., representing things as subjects of our conceptions.

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra."

The words "and," "or" are analogous to the sign $+$, respectively. If x represent "men," y "women," z "European," one has $z(x+y)=zx+zy$, "European men and women" being the same as "European men and European women" (p. 33). Also, $x^2=x$ for "to say 'good, good,' in relation to any subject, though a cumbrous and useless pleonasm, is the same as to say 'good' " (p. 32). The equation $x^2=x$ has no other roots than 0 and 1. "Let us conceive, then, of an Algebra in which the symbols x, y, z, etc., admit indifferently of the values 0 and 1, and of these values alone" (p. 37).

"If x represent any class of objects, then will $1-x$ represent the contrary or supplementary class of objects" (p. 48). The equation $x(1-x)=0$ represents the "principle of contradiction" (p. 49). "The operation of division cannot be *performed* with the symbols with which we are now engaged. Our resource, then, is to *express* the operation, and develop the results by the method of the preceding chapter (p. 89). We may properly term $\frac{0}{0}$ an *indefinite class symbol,* . . . . *some, none,* or *all* of its members are to be taken" (p. 90, 92). A coefficient $\frac{1}{0}$ shows that the constituent to which it belongs must be equated to zero (p. 91). C. I. Lewis remarks that Boole's methods of solution sometimes involve an uninterpretable stage.[3]

[1] C. I. Lewis, *op. cit.*, p. 52.

[2] George Boole, *Laws of Thought* (London and Cambridge, 1854), p. 27.

[3] C. I. Lewis, *op. cit.*, p. 57.

679. *Signs of Jevons.*—Jevons[1] endeavored to simplify Boole's algebra by discarding what has no obvious interpretation in logic. But Jevons, in the opinion of C. I. Lewis,[2] "unduly restricts the operations and methods of Boole." Jevons discards the inverse operations $A-B$ and A/B, and he interprets the sum of A and B as "either A or B, where A and B are not necessarily exclusive classes." In 1864 he uses the notation $A+B$, later[3] he uses $A\cdot|\cdot B$. Thus $A+B=A$ with Jevons, while with Boole $A+B$, is not interpretable as any relation of logical classes.

680. *Remarks of Macfarlane.*—In continuation of the researches of Boole, De Morgan, and Jevons was Alexander Macfarlane's *Principles of the Algebra of Logic* (Edinburgh, 1879), from which (p. 32) we quote: "The reason why Formal Logic has so long been unable to cope with the subtlety of nature is that too much attention has been given to *pictorial notations.* Arithmetic could never be developed by means of the Roman system of notations; and Formal Logic cannot be developed so long as Barbara is represented by

$$\underline{C,\ \blacktriangleleft\ :M,\ \blacktriangleleft\ :T\ ;}$$

or even by the simpler spicular notation of De Morgan. We cannot manipulate data so crudely expressed; because the nature of the symbols has not been investigated, and laws of manipulation derived from their general properties."

681. *Signs of C. S. Peirce.*—C. S. Peirce made numerous contributions to symbolic logic, advancing the work of Boole and De Morgan. His researches contain anticipations of the important procedures of his successors. Among the symbols that C. S. Peirce used in 1867[4] are "the sign of equality with a comma beneath," to express numerical identity; thus $a \underset{,}{=} b$ indicates that a and b denote the same class; $a+,b$ denoting all the individuals contained under a and b together; $a,\ b$ denoting "the individuals contained at once under the classes a and b," $\bar{a}$ denoting *not-a.*, $[1;0]$ standing for "some uninterpretable symbol"; $a;b$ being a logically divided by b; $a:b$ "being the maximum value of $a;b$"; $\underset{,}{-}$ being the sign of logical subtraction.

In 1867 C. S. Peirce gave a "Description of a Notation for the

[1] W. S. Jevons, *Pure Logic* (London, 1864).

[2] C. I. Lewis, *op. cit.*, p. 73.

[3] See C. I. Lewis, *op. cit.*, "Bibliography of Symbolic Logic," p. 395.

[4] C. S. Peirce, "On an Improvement in Boole's Calculus of Logic," *Proceedings of the American Academy of Arts and Sciences*, Vol. VII (Cambridge and Boston, 1868), p. 250–61.

Logic of Relatives"[1] which contains an enlarged and somewhat modified array of symbols. He lets $\prec$ stand for "inclusion in" or "being as small as"; $x \dotplus, y$ for addition, where the comma is written except when the inverse operation (subtraction) is determinative. In his logic of relatives multiplication xy is not generally commutative. Commutative multiplication is written with a comma, $x,\ y=y,\ x$. Invertible multiplication is indicated by a dot: $x.y$. He lets (p. 319) log *abc* † *def* ‡ *ghi* stand for (log *abc*)*ghi*, the base being *def*. Indeterminative subtraction (p. 320) is marked $x \,\overline{,}\, y$. The two divisions are indicated as in $(x:y)y=x$, $x\dfrac{y}{x}=y$. The two inverses of involution are as in $(\sqrt[x]{y})^x=y$, $x^{\log_x y}=y$. On page 321 Peirce gives his father's notation ɕ for $(1+i)^{\frac{1}{i}}$, where i is an infinitesimal, ∂ for 3.14159 . . . . , ⌡ for $\sqrt{-1}$. He denotes (p. 322) absolute terms by the Roman alphabet, relative terms by italics, and conjugate terms by a kind of type called Madisonian. He denotes also individuals by capitals, generals by small letters, general symbols for numbers in black letter, operations by Greek letters, the number of a logical term by [*t*]. In multiplication (p. 326) "identical with" is *1*, so that $x1=x$ ("a lover of something identical with anything is the same as a lover of that thing"). On page 326 he lets $\mathfrak{g}xy$ signify "a giver to x of y." He uses (p. 328) the marks of reference † ‡ ‖ § ¶, which are placed adjacent to the relative term and before and above the correlate of the term. "Thus, giver of a horse to a lover of a woman may be written,

$$\mathfrak{g}\dagger\ddagger\,{}^{\dagger}l_{\|}\,{}^{\|}w\,{}^{\ddagger}h\text{.''}$$

Lack of space prevents the enumeration here of the symbols for infinitesimal relatives.

682. Two pupils of C. S. Peirce, Miss Christine Ladd (now Mrs. Fabian Franklin) and O. H. Mitchell, proposed new notations. Peirce[2] indicated by "Griffin $\prec$ breathing fire" that "no griffin not breathing fire exists" and by "Animal $\overline{\prec}$ Aquatic" that a non-aquatic animal does exist; Peirce adds: "Miss Ladd and Mr. Mitchell also use two signs expressive of simple relations involving existence and non-existence; but in their choice of these relations they diverge

[1] *Memoirs Amer. Acad. of Arts and Sciences* (Cambridge and Boston; N.S.), Vol. IX (1867), p. 317–78.

[2] *Studies in Logic*, by members of the Johns Hopkins University (Boston, 1883), p. iv.

both from McColl and me, and from one another. In fact, of the eight simple relations of terms signalized by De Morgan, Mr. McColl and I have chosen two, Miss Ladd two others, Mr. Mitchell a fifth and sixth." Mrs. Ladd-Franklin[1] uses the copula $\vee$, so that $A \vee B$ signifies "A is in part B," and $A \overline{\vee} B$ signifies "A is excluded from B."

683. O. H. Mitchell[2] showed how to express universal and particular propositions as Π and Σ functions. "The most general proposition under the given conditions is of the form $\Pi(F_u + \Sigma G_1)$ or $\Sigma(F, \Pi G_u)$, where F and G are any logical polynomials of class terms, Π denotes a product, and Σ denotes a sum."

684. *Signs of R. Grassmann.*—Robert Grassmann, the brother of Hermann Gunther Grassmann of the Ausdehnungslehre, wrote on logic and the logical development of elementary mathematics. He develops the laws of classes, but his development labors under the defect, as C. S. Peirce observed, of breaking down when the number of individuals in the class is infinite.[3]

In R. Grassmann's *Formenlehre*[4] of 1872 he uses as the general sign of combination (*Knüpfung*) a small circle placed between the two things combined, as $a \circ b$. Special signs of *Knüpfung* are the sign of equality ($=$), and the sign of inequality which he gives as $\gtrless$. When two general signs of combination are needed, he uses[5] $\circ$ and $\odot$. The notation[6] $\bar{a}$ means "not-a." Accordingly, a double negation $\bar{\bar{a}}$ gives a. Instead of the double sign $\leqq$ he introduces the simple sign[7] $\angle$, so that $a \angle u$ means that a is either equal to u or subordinate to it.

685. *Signs of Schröder.*—Ernst Schröder, of Karlsruhe, in 1877 published investigations on Boole's algebra and later his advances on C. S. Peirce's researches. Peirce's original ideas crossed the Atlantic to Continental Europe, but not his symbolism. It was Schröder who gave to the algebra of logic "its present form,"[8] in so far as it may be said to have any universally fixed form. In 1877 Schröder used largely the symbolism of Boole. If in $c+b=a$ and $cb=a$ the inverse operations are introduced by solving each of the two equations for the c,

[1] *Op. cit.*, p. 25.

[2] *Op. cit.*, p. 79; see also C. I. Lewis, *op. cit.* (1918), p. 108.

[3] C. I. Lewis, *op. cit.*, p. 107.

[4] Robert Grassmann, *Formenlehre oder Mathematik* (Stettin, 1872), p. 8.

[5] *Op. cit.*, p. 18.

[6] Robert Grassmann, *Begriffslehre der Logik* (Stettin, 1872), p. 16, 17.

[7] *Op. cit.*, p. 24.

[8] C. I. Lewis, *op. cit.*, p. 118.

and the expressions $a-b$ and $a:b=\frac{a}{b}$ receive complete and general solutions, he denotes them by $c=a\div b$ and $c=a::b$, respectively.[1]

Later Schröder used[2] $\subset$ for "is included in" (*untergeordnet*) and $\supset$ for "includes" (*übergeordnet*), $\Subset$ "included in or equal," in preference to the symbols $<$, $>$ and $\leqq$ used by some authors on logic, (it was introduced by Schröder[3] in 1873, three years after C. S. Peirce's $\prec$). Schröder employs also $\not\supseteq$ (p. 167). Later, Couturat rejected Schröder's $\Subset$, "parce qu'il est complexe tandis que la relation d'inclusion est simple,"[4] and uses $<$.

Schröder represents the identical product and sum (logical aggregate) of a and b in his logic by ab (or $a.b$) and $a+b$. If a distinction is at any time desirable between the logical and arithmetical symbols, it can be effected by the use of parentheses as in $(+)$, $(\cdot)$ or by the use of a cross of the form $\cdot\!\!+\!\!\cdot$, or of smaller, black, or cursive symbols. Schröder criticizes C. S. Peirce's logical symbols, $=$, $+$, $-$ with commas attached, as rather clumsy.[5] Schröder represents the "identical one" by *1*, while Jevons designated it the "Universe U" and Robert Grassmann the "*Totalität T*." Peirce had used *1* in his earlier writings, but later[6] he and his pupils used ∞, as did also W. Wundt.[7] Schröder[8] justifies his notation partly by the statement that in the theory of probability, as developed by De Morgan, Boole, C. S. Peirce, Macfarlane, and McColl, the identical one corresponds always to the symbol, *1*, of certainty. Schröder lets a_1 stand[9] for "not-a"; Boole, R. Grassmann, and C. S. Peirce had represented this by $\bar{a}$, as does later also Schröder himself.[10] Couturat[11] rejects $\bar{a}$ as typographically inconvenient; he rejects also Schröder's a_1, and

[1] Ernst Schröder, *Der Operationskreis des Logikkalkuls* (Leipzig, 1877), p. 29.

[2] E. Schröder, *Vorlesungen über die Algebra der Logik*, Vol. I (Leipzig, 1890), p. 129, 132.

[3] Schröder, *Vorlesungen*, Vol. I, p. 140, 712.

[4] Louis Couturat, "L'Algèbre de la logique," *Scientia* (Mars, 1905), No. 24, p. 5.

[5] Schröder, *op. cit.*, p. 193.

[6] C. S. Peirce in *Amer. Jour. of Math.*, Vol. III (1880), p. 32. See Schröder, *op. cit.*, p. 274.

[7] W. Wundt, *Logik*, Vol. I (1880).

[8] Schröder, *op. cit.*, p. 275.

[9] Schröder, *op. cit.*, p. 300, 301.

[10] E. Schröder, *Algebra und Logik der Relative*, Vol. I (Leipzig, 1895), p. 18, 29.

[11] Louis Couturat, *op. cit.* (Mars, 1905), No. 24, p. 22.

adopts McColl's symbol a' as not excluding the use of either indices or exponents. In his logic of relatives Schröder[1] let i and j represent a pair of ordered elements in his first domain; they are associated as binary relatives by the symbolism $i:j$, and he introduces also the identical moduli 1, 0 and the relative moduli 1', 0'. The relative product[2] is $a;b$; the relative sum $a \jmath b$; the converse of a is $\breve{a}$.

686. *Signs of MacColl.*—Hugh MacColl worked for many years quite independently of Peirce and Schröder. His symbolic logic of 1906 contains a notation of his own. His sign[3] of equivalence is $=$, of implication is $:$, of disjunction or alternation is $+$, of a proposition is A^B where A is the subject and B the predicate, of non-existence is O. MacColl lets $A^B \times C^D$, or $A^B \cdot C^D$, or $A^B C^D$ mean that A belongs to B and that C belongs to D; he lets $A^B + C^D$ mean that either A belongs to B or else C to the class D. He classifies statements not simply as "true" or "false," but as "true," "false," "certain," "impossible," "variable," respectively, denoted by the five Greek letters[4] τ, ι, ϵ, η, θ. Thus A^η means that A is impossible, $A \therefore B$ means[5] that "A is true, therefore B is true"; $B \because A$ means "B is true because A is true"; (AB) denotes the total of individuals common to A and B; (AB') denotes the total number in A but not in B; letting x be any word or symbol, $\phi(x)$ is any proposition containing x; the symbol $\frac{A}{B}$ expresses the chance that A is true on the assumption that B is true."[6]

687. *Signs of Frege.*—Gottlob Frege, of the University of Jena, published several books on the logic of mathematics, one in 1879, another in 1893.[7] Two of them precede Schröder's *Vorlesungen*, yet Frege's influence on Schröder was negligible. Nor did Peano pay attention to Frege, but Bertrand Russell was deeply affected by him. This early neglect has been attributed to Frege's repulsive symbolism.

[1] Schröder, *op. cit.* (1895), p. 8, 24–36.

[2] Schröder, *loc. cit.* (1895), p. 30.

[3] Hugh MacColl, *Symbolic Logic and Its Applications* (London, 1906), p. 4, 8. One of his early papers was on "Symbolical or Abridged Language, with an Application to Mathematical Probability," *Mathematical Questions . . . . from the Educational Times*, Vol. XXVIII (1878), p. 20, 100. Before about 1884 the name was spelled "McColl."

[4] MacColl, *Symbolic Logic*, p. 6.

[5] MacColl, *op. cit.*, p. 80, 94, 96.

[6] MacColl, *op. cit.*, p. 128.

[7] G. Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Halle, 1879); G. Frege, *Grundgesetze der Arithmetik* (2 vols.; Jena, 1893–1903).

In a paper on the "Fundamental Laws of Arithmetic,"[1] Frege himself admits: "Even the first impression must frighten people away: unknown signs, pages of nothing but strange-looking formulas. It is for that reason that I turned at times toward other subjects." One of the questions which Frege proposed to himself was (as P. E. B. Jourdain expressed it) "as to whether arithmetical judgments can be proved in a purely logical manner or must rest ultimately on facts of experience. Consequently, he began by finding how far it was possible to go in arithmetic by inferences which depend merely on the laws of general logic. In order that nothing that is due to intuition should come in without being noticed, it was most important to preserve the unbrokenness of the chain of inferences; and ordinary language was found to be unequal to the accuracy required for this purpose."[2]

In Figure 125 he starts out to prove the theorem: "The number of a concept is equal to the number of a second concept, if a relation images (*abbildet*) the first upon the second and the converse of that relation images the second upon the first."

In the publication of 1893 Frege uses all the symbols employed by him in 1879, except that he now writes $=$ instead of $\equiv$. He uses $\vdash$ to indicate that a statement is true. Thus[3] the symbolism $\vdash 2^2=4$ declares the truth of the assertion that $2^2=4$. He distinguishes between the "thought" and the "judgment"; the "judgment" is the recognition of the truth of the "thought." He calls the vertical stroke in $\vdash$ the "judgment stroke" (*Urtheilstrich*); the horizontal stroke is usually combined with other signs but when appearing alone it is drawn longer than the minus sign, to prevent confusion between the two. The horizontal stroke is a "function name" (*Functionsname*) such that $-\Delta$ is true when Δ is true; and $-\Delta$ is false when Δ is false (p. 9).

The value of the function $\top\xi$ shall be false for every argument for which the value of the function $-\xi$ is true, and it shall be true for all other arguments. The short vertical stroke (p. 10) in $\top$ is a sign of negation. $\top\Delta$ means the same as $\top(-\Delta)$, the same as $-\top\Delta$, and the same as $-\dot{\top}(-\Delta)$.

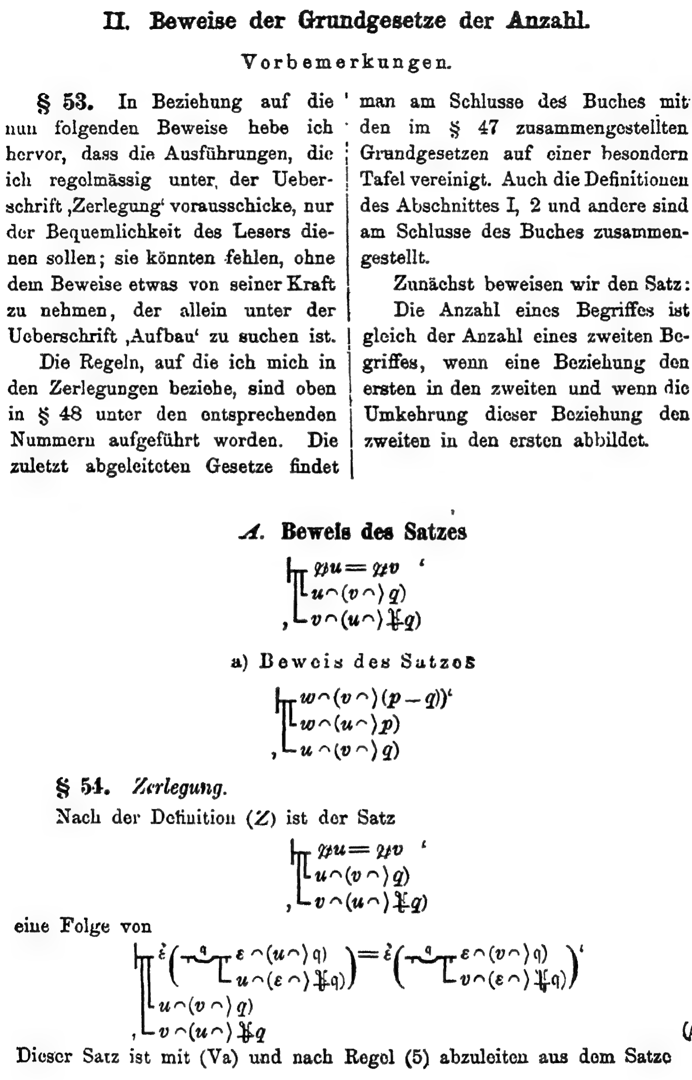
The notation (p. 12) $-\!\overset{\mathfrak{a}}{\smile}\!\top 2+3.a=5.a$ declares the generality of negation, i.e., it declares as a truth that for every argument the value

[1] *Monist*, Vol. XXV (Chicago, 1915), p. 491.

[2] P. E. B. Jourdain, *Monist*, Vol. XXV (1915), p. 482.

[3] G. Frege, *Grundgesetze der Arithmetik*, Vol. I. p. 9.

of the function $\smile 2+3.\xi=5.\xi$ is true. The notation $\neg\!\smile\!\!-\; 2+3.a=5a$ declares the negation of the generality, i.e., it signifies the truth that



II. Beweise der Grundgesetze der Anzahl.

Vorbemerkungen.

§ 53. In Beziehung auf die nun folgenden Beweise hebe ich hervor, dass die Ausführungen, die ich regelmässig unter der Ueberschrift ‚Zerlegung' vorausschicke, nur der Bequemlichkeit des Lesers dienen sollen; sie könnten fehlen, ohne dem Beweise etwas von seiner Kraft zu nehmen, der allein unter der Ueberschrift ‚Aufbau' zu suchen ist.

Die Regeln, auf die ich mich in den Zerlegungen beziehe, sind oben in § 48 unter den entsprechenden Nummern aufgeführt worden. Die zuletzt abgeleiteten Gesetze findet man am Schlusse des Buches mit den im § 47 zusammengestellten Grundgesetzen auf einer besondern Tafel vereinigt. Auch die Definitionen des Abschnittes I, 2 und andere sind am Schlusse des Buches zusammengestellt.

Zunächst beweisen wir den Satz:

Die Anzahl eines Begriffes ist gleich der Anzahl eines zweiten Begriffes, wenn eine Beziehung den ersten in den zweiten und wenn die Umkehrung dieser Beziehung den zweiten in den ersten abbildet.

A. Beweis des Satzes

$$\vdash\!\!\top\; \mathfrak{N}u = \mathfrak{N}v \;\; \llcorner u \frown (v \frown) q) \;\; \llcorner v \frown (u \frown) \mathfrak{X} q)$$

a) Beweis des Satzes

$$\vdash\!\!\top\; w \frown (v \frown)(p - q)) \;\; \llcorner w \frown (u \frown) p) \;\; \llcorner u \frown (v \frown) q)$$

§ 54. *Zerlegung.*

Nach der Definition (Z) ist der Satz

$$\vdash\!\!\top\; \mathfrak{N}u = \mathfrak{N}v \;\; \llcorner u \frown (v \frown) q) \;\; \llcorner v \frown (u \frown) \mathfrak{X} q)$$

eine Folge von

$$\vdash\!\!\top\; \grave{\varepsilon}\left(\smile^{\mathfrak{q}}\!\!\top \varepsilon \frown (u \frown) \mathfrak{q}) \;\llcorner u \frown (\varepsilon \frown) \mathfrak{X}\mathfrak{q})\right) = \grave{\varepsilon}\left(\smile^{\mathfrak{q}}\!\!\top \varepsilon \frown (v \frown) \mathfrak{q}) \;\llcorner v \frown (\varepsilon \frown) \mathfrak{X}\mathfrak{q})\right) \;\; \llcorner u \frown (v \frown) q) \;\; \llcorner v \frown (u \frown) \mathfrak{X} q \qquad (\beta$$

Dieser Satz ist mit (Va) und nach Regel (5) abzuleiten aus dem Satze

FIG. 125.—Frege's notation as found in his *Grundgesetze* (1893), Vol. I, p. 70

not for every argument the value of the function $2+3.\xi=5.\xi$ is true.

Frege (p. 20) lets T^{ξ}_{ζ} be a function with two arguments whose value is false when the ζ-argument is assumed to be true and the ξ-argument is assumed to be false; in all other cases the functional value is true. The vertical stroke is here designated as a "condition stroke" (*Bedingungstrich*).

$\urcorner\!\mathsf{T}^{\Gamma}_{\Delta}$ is true when, and only when, Δ is true and Γ is not true.

Hence, $\vdash\!\mathsf{T}^{2>3}_{2+3=5}$ affirms the statement that "2 is not greater than 3, and the sum of 2 and 3 is 5" (p. 21).

$\vdash\!\mathsf{T}^{3>2}_{2+3=5}$ affirms the statement that "3 is greater than 2, and the sum of 2 and 3 is 5."

$\vdash\!\mathsf{T}^{2^3=3^2}_{1^2=2^1}$ affirms the statement that "Neither is the 3rd power of 2 the 2nd power of 3, nor is the 2nd power of 1 the 1st power of 2."

One can pass (p. 27) from the proposition $\vdash\!\mathsf{T}^{\Gamma}_{\Delta}$ to $\vdash\!\mathsf{T}^{\Delta}_{\Gamma}$, and vice versa. These two passages are symbolized in this manner:

$$\begin{array}{c} \vdash\!\mathsf{T}^{\Gamma}_{\Delta} \\ \times \\ \vdash\!\mathsf{T}^{\Delta}_{\Gamma} \end{array} \qquad \text{and} \qquad \begin{array}{c} \vdash\!\mathsf{T}^{\Delta}_{\Gamma} \\ \times \\ \vdash\!\mathsf{T}^{\Gamma}_{\Delta} \end{array}$$

688. *Signs of Peano.*—Schröder once remarked that in symbolic logic we have elaborated an instrument and nothing for it to do. Peano proceeded to use this instrument in mathematical proof. In 1888 he published his *Calcolo geometrico*,[1] where he states that, in order to avoid confusion between logical signs and mathematical signs, he substitutes in place of the signs $\times$, $+$, A_1, 0, 1, used by Schröder[2] in 1877, the signs $\frown$, $\smile$, $-A$, $\bigcirc$, $\bullet$.

In his *Formulaire de mathématiques*, the first volume of which appeared at Turin in 1895, Peano announces confidently a realization of the project set by Leibniz in 1666, namely, the creation of a universal script in which all composite ideas are expressed by means of conventional signs of simple ideas, according to fixed rules. Says Peano:[3] "One may change the shape of the signs, which have been

[1] G. Peano, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann* (Torino, 1888), p. x.

[2] E. Schröder, *Der Operationskreis der Logikkalkuls* (1877).

[3] *Formulaire de mathématiques*, Tome I (Turin, 1895), Introd. (dated 1894), p. 52.

introduced whenever needed; one may suppress some or annex others, modify certain conventions, etc., but we are now in a position to express all the propositions of mathematics by means of a small number of signs, having a precise signification and subject to well determined rules."

The *Formulaire de mathématiques* of 1895 was not the earliest of Peano's publications of this subject. The first publication of the *Formulaire de logique* was in 1891, in the *Rivista di matematica* (p. 24, 182). This was reproduced with additions as Part I of the *Formulaire de mathématiques*, Volume I (1893–95). The historical statements in it are due mainly to G. Vailati. The first three parts of the first volume deal with mathematical logic, algebraic operations, arithmetic, and were prepared by Peano with the aid of G. Vailati, F. Castellano, and C. Burali-Forti. The subsequent parts of this first volume are on the theory of magnitudes (by C. Burali-Forti), of classes (by Peano), of assemblages (by G. Vivanti), of limits (by R. Bettazzi), of series (by F. Giudice), of algebraic numbers (by G. Fano).

689. Peano expresses the relations and operations of logic in Volume I of his *Formulaire de mathématiques* (Introduction, p. 7) by the signs ϵ, c, $\supset$, $=$, $\frown$, $\smile$, $-$, V, Λ, the meanings of which are, respectively, "is" (i.e., is a number of), "contains," "is contained in," "is equal" (i.e., is equal to), "and," "or," "not" "all," "nothing." He adds that c and v are mentioned here for the sake of symmetry, but are not used in practice. These symbols are used for propositions as well as for classes, but receive a somewhat different signification when applied to propositions. For example, if a and b are classes, $a \supset b$ signifies "a is contained in b," but if a and b are propositions, then $a \supset b$ signifies "from a one deduces b" or "b is a consequence of a." As examples of the notation used for certain classes K, we quote (Introduction, p. 4):

"N	signifies	number, integral positive
n	"	number, integral (positive, zero or negative)
N_0	"	number, integral positive or zero
R	"	number, rational positive
r	"	number, rational
Q	"	number, real positive (quantity)
q	"	number, real
Q_0	"	number, real positive or zero."

As symbols of aggregation, dots are used here in preference to parentheses.

On pages 134–39 of the first volume of his *Formulaire* Peano gives a table of the signs used in that volume. There are altogether 158 signs, of which 26 are symbols having special forms (ideographs), 16 are Greek letters, and 116 are Latin letters or combinations of them.

As specimens of the use he makes of his symbols, we give the following:

Vol. I, p. 1. $a \supset b . b \supset c . \supset . a \supset c$ [Pp.].

"If a is contained in b, and b in c, then, a is contained in c."

p. 16. $a \epsilon Q . x \epsilon q . \supset . a^x \epsilon Q$.

Primitive proposition: "If a is a positive real number, and x is a real number, then a^x is a positive real number."

p. 49. $A, B, C, D, U, U' \epsilon G . \supset : A|U \epsilon Q$.

"If A, B, C, D, U, U' are magnitudes (G=grandeur), then the fraction $A|U$ is a positive real number."

p. 85. $u, v . \epsilon qfN . m \epsilon N . \supset : \Sigma u_\infty \epsilon q . \supset . \lim u_n = 0. \quad \lim \sum_n^{n+m} u = 0.$

"If u is a real number defined in terms of positive integral numbers, and m is a positive integer, then, when the sum of an infinite number of successive u's is a real number, it follows that the limit of u_n is zero. and the limit of the sum of the u's from u_n to u_{n+m} is also zero."

Volume II of Peano's *Formulaire de mathématiques* appeared at Turin in three numbers: No. 1 in 1897, No. 2 in 1898, and No. 3 in 1899. Number 1 gives a fuller development of his symbolism of logic. Additional symbols are introduced and a few of the older notations are modified.

One of the new symbols is $\exists$ (No. 1, p. 47), $\exists a$ signifying "il y a des a" or "there are a's." It was introduced "because the notation $a - = \Lambda$ was held by many of the collaborators as long and differing too widely from ordinary language." Peano adds: "One has thus the advantage that by the three signs $\exists$, $\frown$, $-$ one can express $\supset$ and Λ [viz. (p. 14, § 1), $a \supset b . = . - \exists(a - b)$ and $a = \Lambda . = . - \exists a.$]."

ιx expresses the class formed by the object x; reciprocally, if a is a class which contains a single individual x, Peano now writes (No. 1, p. 49) $\bar{\iota} a$ to indicate the individual which forms the class a, so that $x = \bar{\iota} a . = . a = \iota x$. In other words, "$x$ is the individual which constitutes the class a" is a statement identical with the statements that "a is the class formed by the object x." Example: $a, b \epsilon N . b > a . \supset . b - a = \bar{\iota} N \frown \overline{x \epsilon}(a + x = b)$ signifies: "If a and b are positive integers, and $b > a$,

it follows that $b-a$ is the number which is to be added to a, to obtain b." This class exists and contains the single individual $b-a$.

Among other changes, Peano extends the use of the sign ‘. In the first volume[1] he had used $\smile$‘ for "the smallest class containing all the classes u," $\frown$‘ for "the greatest class contained in all the classes $\smile$." Now he lets (No. 1, p. 51) K‘a stand for "the class contained in a."

In grouping symbols Peano no longer limits himself to the use of dots; now he uses also parentheses.

690. Number 2 of Volume II of Peano's *Formulaire* is devoted mainly to the theory of numbers, integral, fractional, positive, and negative. Theorems are expressed in ideographic signs, together with proofs and historical notes. Even the references to contributors are indicated by special symbols. Thus $\sqcup$ means "Peano, *Arithmeticae principia*, 1889"; $\lhd$ means "Vacca, *Form. de Math.*, t. II."

Some additional symbols are introduced; others are altered in form. Thus ɔ becomes $\supset$. By the symbolism $p.\supset x\ldots.z.q$ is expressed "from p one deduces, whatever $x\ldots.z$ may be, and q."

The new symbolism $a/\!\!\!\!\!\! \smallfrown m$ signifies a^m. Peano says on page 20: "It permits one to write all the signs in the same line, which presents a great typographical advantage in complicated formulas." It is the radical sign $\surd$ turned in its own plane through 180°.

Number III of Volume II of Peano's *Formulaire* (Turin, 1899) contains, as stated in its Preface, "the principal propositions thus far expressed by ideographic symbols. It is divided into §§. Each § has for its heading an ideographic sign. These signs follow each other in an order so that all signs appear defined by the preceding (with the exception of primitive ideas). Every § contains the propositions which one expresses by the sign heading that § and those preceding. .... This disposition, already partially applied in the preceding publications, is followed here with greater rigor, ..." Some new notations are given. For instance (No. 3, p. 23, § 8), the new symbol $\vdots$ is defined thus: "$(u \vdots v)$ signifies the assemblage of couples formed by an object of the class u with an object of the class v; this new sign is to be distinguished from $(u\,;v)$ which signifies the couple of which the two elements are the classes u and v."

$$u,\ v\epsilon Cls\cdot\supset\cdot(u \vdots v)=(x\,;y)\,\mathcal{3}\,(x\epsilon u\cdot y\epsilon v)\ \text{Df.}$$

This ideographic definition is in words: "If u and v are classes, then $(u \vdots v)$ is equal to the couple of which the two elements are x and

[1] *Formulaire de mathématiques*, Vol. I, Introd., p. 25, § 21.

y which satisfy the condition, x is u, and y is v." The symbol ꜿ may be read "which."

An edition (Vol. III) of the *Formulaire de mathématiques* of the year 1901 was brought out in Paris.

691. To give an idea of the extent to which the Peano ideographs have been used, it is stated in the seventh volume of the *Revue de mathématiques* (1901) that the symbols occur in sixty-seven memoirs published in different countries by fifteen authors. The third volume of the *Formulaire* collects the formulae previously published in the *Revue de mathématiques* and in the *Formulaire*, together with new propositions reduced to symbols by M. Nassò, F. Castellano, G. Vacca, M. Chini, T. Boggio.

In Volume IV of the *Formulaire* (Turin, 1903) seven pages are given to exercises on mathematical logic and drill in the mastery of the symbols, and seventeen pages to biographical and bibliographical notes prepared by G. Vacca.

Volume V of the *Formulaire*, the final volume which appeared at Turin in 1908, is much larger than the preceding, and is written in Latin instead of French. It gives full explanations of symbols and many quotations from mathematical writers. It is divided into eight parts devoted, respectively, to "Logica-Mathematica," "Arithmetica," "Algebra," "Geometria," "Limites," "Calculo differentiale," "Calculo integrale," "Theoria de curvas."

Peano compared[1] his symbols with those of Schröder in the following grouping:

Schröder.........	0	1	$+$	$\cdot$	Σ	Π	$\bar{a}$	$\leqq$
Formulario.......	Λ	V	$\smile$	$\frown$	$\smile$‘	$\frown$‘	$-a$	$\epsilon\supset$

692. *Signs of A. N. Whitehead.*—A. N. Whitehead, in his *Treatise on Universal Algebra*, Volume I (Cambridge, 1898), deviates considerably from the Peano symbolism, as appears from the following comparison, made by Vacca,[2] of the symbols for addition, none, multiplication, the "universe," and "supplementary element":

Whitehead...............	$a+b$,	0,	ab,	i,	$\bar{a}$
Peano....................	$a\smile b$,	Λ,	ab,	V,	$-a$

693. *Signs of Moore.*—Eliakim Hastings Moore, in his General Analysis, introduces a thorough uniformization of notations, but displays conservatism in the amount and variety of symbolism used. He

[1] *Revue de mathématiques* (Turin), Vol. VI (1896–99), p. 96.

[2] G. Vacca, *Revue de mathématiques*, Vol. VI (Turin, 1896–99), p. 102.

feels that the extreme tendencies of Peano, Schroeder, Whitehead, and Russell are not likely to be followed by the working mathematicians, that, on the other hand, writers in the abstract regions of mathematics will find that mathematical and logical ideas of constant recurrence can profitably be indicated symbolically, in the interest of brevity and clarity. Moore does not aim to reduce all mathematical representation and reasoning to ideographic notations; he endeavors to restrict himself to such symbolism as will meet the real needs of the great body of mathematicians interested in recent developments of analysis.

In general, Moore uses small Latin letters for elements, capital Latin letters for transformations or properties or relations, Greek letters for functions, and capital German letters for classes. For example, if a property P is defined for elements t of a class $\mathfrak{T}$, the notation t^P concerning an element t indicates that t has the property P. Similarly, $t_1 Rt_2$ or $(t_1, t_2)^R$ indicates that the element t_1 is in the relation R to the element t_2. Naturally, t^P in the hypothesis of an implication indicates, "for every element t having the property P." In this notation Moore follows more or less closely Hugh MacColl.[1]

Consider two classes $\mathfrak{P} \equiv [p]$, $\mathfrak{Q} \equiv [q]$ of elements p, q. A single-valued function φ on $\mathfrak{P}$ to $\mathfrak{Q}$, $\varphi^{\text{on } \mathfrak{P} \text{ to } \mathfrak{Q}}$, is conceptually a table giving for every argument value p a definite functional value, $\varphi(p)$ or q_p, belonging to the class $\mathfrak{Q}$. The function φ as the table of functional values $\varphi(p)$ has the notation $\varphi \doteq (\varphi(p) \mid p) \equiv (q_p \mid p)$, where the "$\mid p$" indicates that p ranges over $\mathfrak{P}$. This precise notational distinction between function φ and functional value $\varphi(p)$ is of importance: it obviates the necessity of determining from the context whether $\varphi(p)$ denotes the function φ or the functional value $\varphi(p)$ for the value p of the argument. "Function φ on $\mathfrak{P}$ to $\mathfrak{Q}$" is, of course, precisely Georg Cantor's[2] "Belegung von $\mathfrak{P}$ mit $\mathfrak{Q}$"; for use in analysis the locution "function on to" is happier than "Belegung von mit."

694. We quote the list of logical signs which E. H. Moore used in his *General Analysis* in 1910. With the exception of $\doteq$; $\equiv$; $\sim$; [ ], these signs are taken from Peano's *Formulario* of 1906 and are used approximately in the sense of Peano. The list is as follows:[3]

[1] Hugh MacColl in *Bibliothèque du Congrès Internationale de Philosophie. III. Logique et Histoire* (Paris, 1901), p. 137, 167.

[2] G. Cantor, *Mathematische Annalen*, Vol. XLVI (1895), p. 486.

[3] E. H. Moore, *Introduction to a Form of General Analysis* (New Haven, 1910; The New Haven Mathematical Colloquium), p. 150. See also *Bull. Amer. Math. Society* (2d ser.), Vol. XVIII (1912), p. 344, 345; E. H. Moore, "On the Funda-

"$=$ logical identity

$\neq$ logical diversity

$\equiv$ definitional identity

$\cdot\supset\cdot$ (for every . . . .) it is true that
( ) implies ( )
if ( ), then ( )

$\cdot\subset\cdot$ ( ) is implied by ( )

$\cdot\sim\cdot$ ( ) is equivalent to ( )
( ) implies and is implied by ( )

$\supset$; $\subset$; $\sim$ implies; is implied by; is equivalent to (as relations of properties)

$\exists$ there exists a (system; class; element; etc.)

$\ni$ such that; where

$\cdot$ and

. .; : :; .·. .·. signs of punctuation in connection with signs of implication, etc.; the principal implication of a sentence has its sign accompanied with the largest number of punctuation dots

$\smile$ or

$-$ not

[ ] a class of (elements; functions; etc.)

[all] the class consisting of all (elements; functions; etc., having a specified property or satisfying a specified condition)

$\mathsf{U}[\mathfrak{P}]$ the least common superclass of the classes $\mathfrak{P}$ of the class $[\mathfrak{P}]$ of classes

$\cap[\mathfrak{P}]$ the greatest common subclass of the classes $\mathfrak{P}$ of the class $[\mathfrak{P}]$ of classes."

In further illustration of Moore's use of logical symbols, we quote the following:

"From a class: $[P]$, of properties of elements arises by composition

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mental Functional Operation of a General Theory of Linear Integral Equations," *Proceedings of the Fifth International Congress of Mathematicians* (Cambridge, 1912); E. H. Moore, "On a Form of General Analysis with Applications to Linear Differential and Integral Equations," *Atti del IV Congresso internazionale dei matematici* (Roma, 1909); E. H. Moore, "Definition of Limit in General Integral Analysis," *Proceedings of the National Academy of Sciences*, Vol. I (1915), p. 628–32; E. H. Moore, "On Power Series in General Analysis," *Festschrift David Hilbert zu seinem sechzigsten Geburtstag* (Berlin, 1922), p. 355–64, and *Mathematische Annalen*, Vol. LXXXVI (1922), p. 30–39; E. H. Moore and H. L. Smith, "A General Theory of Limits," *American Journal of Mathematics*, Vol. XLIV (1922), p. 102–21.

the *composite property:* $\cap[P]$, viz., the property of having every property P of the class $[P]$ of properties. . . . ."[1]

"From a class: $[P]$, of properties of elements arises by disjunction the *disjunctive property:* $\cup[P]$, viz., the property of having at least one of the properties P of the class $[P]$ of properties. The notation is so chosen that the class of all elements having the disjunctive property $\cup[P]$ is the class $\cup[\mathfrak{P}]$, viz., the least common superclass of the classes $\mathfrak{P}$ of the corresponding class $[\mathfrak{P}]$ of classes of elements."[2]

"We may write the definitions as follows:

$$x^{P_1 \ldots\ldots P_n} : \equiv : x \ni i^{1 \leqq i \leqq n} . \supset . x^{P_i} ,$$

viz., an element x having the composite property $P_1 \ldots\ldots P_n$ *is by definition* ($\equiv$) an element x *such that* ($\ni$) for every integer i having the property: $1 \leqq i \leqq n$, *it is true that* ($.\supset.$)x has the property P_i;

$$x^{\cap[P]} : \equiv : x \ni P^{[P]} . \supset . x^P ,$$

viz., an element x has the composite property $\cap[P]$ in case for every property P belonging to the class $[P]$ it is true that x has the property P;

$$x^{P_1 \smile P_2 \smile \ldots\ldots \smile P_n} . \equiv . x \ni \exists\, i^{1 \leqq i \leqq n} \ni x^{P_i} ,$$

viz., an element x has the disjunctive property $P_1 \smile \ldots\ldots \smile P_n$ in case *there exists an* ($\exists$) integer i having the property: $1 \leqq i \leqq n$, such that x has the property P_i;

$$x^{\cup[P]} . \equiv . x \ni \exists\, P^{[P]} \ni x^P ,$$

viz., an element x has the disjunctive property $\cup[P]$ in case there exists a property P belonging to the class $[P]$ such that x has the property P."[3]

695. *Signs of Whitehead and Russell.*—The *Principia mathematica* of Alfred North Whitehead and Bertrand Russell, the first volume of which appeared in 1910 at Cambridge, England, is a continuation of the path laid out in recent years mainly by Frege and Peano. Whitehead and Russell state in their Preface: "In the matter of notation, we have as far as possible followed Peano, supplementing his notation, when necessary, by that of Frege or by that of Schröder. A great deal of symbolism, however, has had to be new, not so much through dissatisfaction with the symbolism of others, as through the fact that

[1] E. H. Moore, *Introduction to a Form of General Analysis* (1910), p. 18.

[2] *Op. cit.*, p. 19.

[3] E. H. Moore, *op. cit.*, p. 19, 20.

we deal with ideas not previously symbolised. In all questions of logical analysis, our chief debt is Frege."

In the Introduction, Whitehead and Russell state the three aims they had in view in writing the *Principia:* (1) "the greatest possible analysis of the ideas with which it [mathematical logic] deals and of the processes by which it conducts demonstrations, and at diminishing to the utmost the number of the undefined ideas and undemonstrated propositions"; (2) "the perfectly precise expression, in its symbols, of mathematical propositions"; (3) "to solve the paradoxes which, in recent years, have troubled students of symbolic logic and the theory of aggregates." On matters of notation they state further:

"The use of symbolism, other than that of words, in all parts of the book which aim at embodying strictly accurate demonstrative reasoning, has been forced on us by the consistent pursuit of the above three purposes. The reasons for this extension of symbolism beyond the familiar regions of number and allied ideas are many: (1) The ideas here employed are more abstract than those familiarly considered in language. . . . . (2) The grammatical structure of language is adopted to a wide variety of usages. Thus it possesses no unique simplicity in representing the few simple, though highly abstract, processes and ideas arising in the deductive trains of reasoning employed here. . . . . (3) The adoption of the rules of the symbolism to the processes of deduction aids the intuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed. . . . . (4) The terseness of the symbolism enables a whole proposition to be represented to the eyesight as one whole, or at most in two or three parts." The authors say further: "Most mathematical investigation is concerned not with the analysis of the complete process of reasoning, but with the presentation of such an abstract of the proof as is sufficient to convince a properly instructed mind. For such investigations the detailed presentation of steps in reasoning is of course unnecessary, provided that the detail is carried far enough to guard against error. . . . . In proportion as the imagination works easily in any region of thought, symbolism (except for the express purpose of analysis) becomes only necessary as a convenient shorthand writing to register results obtained without its help. It is a subsidiary object of this work to show that, with the aid of symbolism, deductive reasoning can be extended to regions of thought not usually supposed amenable to mathematical treatment."

The symbolism used in the first volume of their *Principia mathematica* is as follows (1910, p. 5–38):

The small letters of our alphabet are used for variables, except p and s to which constant meanings are assigned in the latter part of the volume;

The capitals A, B, C, D, E, F, I, J, have constant meanings.

Also the Greek letters ϵ, ι, π and, at a later stage, η, θ, and ω have constant meanings;

The letters p, q, r, are called 'propositional letters' and stand for variable *propositions.*

The letters $f\ g\ \phi\ \psi\ \chi\ \theta$ and, at first F, are called 'functional letters' and are used for variable functions.

Other small Greek letters and other ones of our ordinary capitals will be used as special types of variables.

$\sim p$ means *not-p,* which means the negation of p, where p is any *proposition.*

$p \vee q$ means that at least p or q is true; it is the logical sum with p and q as arguments.

$p.q$ means that both p and q are true; it is the logical product with p and q as arguments. Thus, $p.q$ is merely a shortened form for $\sim(\sim p \vee \sim q)$.

$p)q$ means p implies q; it stands for $\sim p \vee q$ when p is true.

$p \equiv q$ means p implies q and q implies p.

$\vdash$ is Frege's "assertion-sign," to show that what follows is asserted. Thus, $\vdash.(p \supset q)$ means an assertion that $p \supset q$ is true; without the assertion-sign, a proposition is not asserted, and is merely put forward for consideration, or as a subordinate part of an asserted proposition.

Dots are used to bracket off propositions, or else to indicate the logical product as shown above. Thus, $p \vee q . \supset . q \vee p$ means 'p or q' implies 'q or p.'

As an example of the authors' "primitive propositions," we quote:

$$\vdash : p \vee q . \supset . q \vee p . \qquad \text{Pp.},$$

i.e., "If p or q is true, then q or p is true. Proposition."

The law of excluded middle is one of the simple propositions and is indicated thus:

$$\vdash . \, p \vee \sim p \, ,$$

i.e., "p is true, or *not-p* is true."

The law of contradiction:

$$\vdash . \sim(p . \sim p) \, ,$$

i.e., "It is not true that p and *not-p* are both true."

The law of double negation:

$$\vdash . p \equiv \sim(\sim p) ,$$

i.e., "p is the negation of the negation of p."

ϕx designates a "propositional function"; it is not a proposition, since x is a variable, having no fixed determined meaning. "x is human" is a propositional function; so long as x remains undetermined, it is neither true nor false.

$(x) . \phi x$ means "$\phi(x)$ is true for all possible values of x."

$(\exists x) . \phi x$ means "there exists an x for which ϕx is true," so that *some* propositions of the range are true.

$(x) . \sim\phi x$ means "there exists no x for which ϕx is true."

$\phi x . \supset_x . \psi x$ is a notation due to Peano and means, "If ϕx is always true, then ψx is true."

$x \epsilon \alpha$ means, as with Peano, "x is α" or "x is a member of the class α."

$\vdash : x \epsilon \hat{z}(\phi z) . \equiv . \phi x$ means " 'x is a member of the class determined by $\phi\hat{z}$' is equivalent to 'x satisfies $\phi\hat{z}$,' or to 'ϕx is true.' " Here $\hat{z}$ is individual z.

$\phi(x, y)$ is a function which determines a *relation* R between x and y.

$\hat{x}\hat{y}\phi(x, y)$ is the relation determined by the function $\phi(x, y)$.

xRy means "x has the relation R to y."

$$\vdash . z\{\hat{x}\hat{y}\phi(x,y)\}w . \equiv . \phi(z,w) ,$$

i.e., " 'z has to w the relation determined by the function $\phi(x,y)$' is equivalent to $\phi(z, w)$."

$\alpha \cap \beta$ is the logical product of two *classes* α and β, representing the class of terms which are members of both.

$$\vdash : x \epsilon \alpha \cap \beta . \equiv . x \epsilon \alpha . x \epsilon \beta ,$$

means " 'x is a member of the logical product of α and β' is equivalent to the logical product of 'x is a member of α' and 'x is a member of β.' "

$\alpha \cup \beta$ is the logical sum of two classes α and β, representing the class of terms which are members of either.

$-\alpha$ is the class of terms of suitable type which are not members of class α; it may be defined thus:

$$-\alpha = \hat{x}(x \sim\epsilon \alpha) \qquad \text{Df.} ,$$

and the connection with the negation of proposition is given by

$$\vdash : x \epsilon - \alpha . \equiv . x \sim\epsilon \alpha .$$

$\alpha \subset \beta$ means inclusion; i.e , class α is contained in class β. As an example of propositions concerning classes which are analogous of propositions concerning propositions, we give:

$$\vdash . \alpha \cap \beta = -(-\alpha \cup -\beta) .$$

The law of absorption holds in the form

$$\vdash : \alpha \subset \beta . \equiv . \alpha = \alpha \cap \beta ,$$

as, for instance, "all Cretans are liars" is equivalent to "Cretans are identical with lying Cretans."

Whitehead and Russell[1] repeat a statement due to Peano, namely, that $x \epsilon \beta$ is not a particular case of $\alpha \subset \beta$; on this point, traditional logic is mistaken.

Definitions and propositions similar to those for classes hold for relations. The symbol for relations is obtained by adding a dot to the corresponding symbol for classes. For example:

$R \dot{\cap} S = \hat{x}\hat{y}(xRy . xSy)$ Df.

$\vdash : x(R \dot{\cap} S)y . \equiv . xRy . xSy,$

i.e., " 'x has to y the relation which is common to R and S' is equivalent to 'x has the relation R to y and x has the relation S to y.' "

$R \dot{\cup} S = \hat{x}\hat{y}(xRy . \mathrm{v} . xSy)$ Df.

$\dot{-} R = \hat{x}\hat{y}\{\sim(xRy)\}$ Df.

$R \mathrel{\dot{\subset}} S . = : xRy . \supset_{x, y} . xSy$ Df.

Further symbols for the algebra of classes:

$\exists ! \alpha$ denotes "α exists."

$\exists ! \alpha . = . (\exists x) . x \epsilon \alpha$ Df.

i.e., " 'α exists' is equivalent to 'x exists and x is α.' "

Λ is a class that has no members (null-class).

V is a universal class, being determined by a function which is always true. Thus, Λ is the negation of V.

The corresponding symbols for relations are:

$\dot{\exists} ! R . = . (\exists xy) . xRy$,

i.e., $\exists ! R$ means that there is at least one couple x, y between which the relation R holds.

$\dot{\Lambda}$ is the relation that never holds.

$\dot{V}$ is the relation that always holds.

$E ! (\iota x)(\phi x) . = : (\exists c) : \phi x . \equiv_x . x = c$ Df.

[1] Whitehead and Russell, *op. cit.*, Vol. I (1910), p. 29.

i.e., " 'the x satisfying $\phi\hat{x}$ exists' is to mean 'there is an object c such that ϕx is true when x is c but not otherwise.' "

$R\text{‘}y = (\iota x)(xRy)$ Df.

The inverted comma may be real "of"; $R\text{‘}y$ is read "the R of y."

For the converse of R, the authors have the notation

$\mathrm{Cnv}\text{‘}\, R = \breve{R} = \hat{x}\hat{y}(yRx)$ Df.

$\overrightarrow{R}\text{‘}y = \hat{x}(xRy)$ Df.

$\overleftarrow{R}\text{‘}x = \hat{y}(xRy)$ Df.

"If R is the relation of parent to child, $\overrightarrow{R}\text{‘}y$ = the parents of y, $\overleftarrow{R}\text{‘}x$ = the children of x."

$D\text{‘}R = \hat{x}\{(\exists y)\cdot xRy\}$ Df.

$D\text{‘}R$ is the *domain* of R, i.e., the class of all terms that have the relation R to y. The *converse domain* of R is symbolized and defined thus:

$\text{Ꭰ}\text{‘}R = \hat{y}\{(\exists x)\cdot xRy\}$ Df.

The sum of the domain and the converse domain is called the "field" and is represented by $C\text{‘}R$,

$C\text{‘}R = D\text{‘}R \cup \text{Ꭰ}\text{‘}R$ Df.

The *relative product* of two relations R and S arises when xRy and ySz; it is written $R \mid S$.

$R \mid S = \hat{x}\hat{z}\{(\exists y)\cdot xRy\cdot ySz\}$ Df.

"Paternal aunt" is the relative product of "sister" and "father."

$R\text{‘‘}\alpha$ is the class of terms x which have the relation R to some member of a class α. We have,

$R\text{‘‘}\alpha = \hat{x}\{(\exists y)\cdot y\epsilon\alpha\cdot xRy\}$ Df.

Thus, if R is *inhabiting* and α is *towns*, then $R\text{‘‘}\alpha$ = inhabitants of towns.

$\iota\text{‘}x$ is the class whose only member is x. Peano and Frege showed that the class whose only member is x is not identical with x.

$\alpha \upharpoonleft R$ is a relation R with its domain limited to members of α (Vol. I [1910], p. 278).

$R \upharpoonright \beta$ is a relation R with its converse domain limited to members of β.

$\alpha \upharpoonleft R \upharpoonright \beta$ is a relation R with both of these limitations.

$\alpha \uparrow \beta$ is the relation between x and y in which x is a member of α and y is a member of β; i.e.,

$\alpha \uparrow \beta = \hat{x}\hat{y}(x\epsilon\alpha \cdot y\epsilon\beta)$ Df.

The foregoing list represents most of the symbols used by Whitehead and Russell in the first volume of their *Principia mathematica*. This volume, after an Introduction of sixty-nine pages, is devoted to

two parts; the first part on "Mathematical Logic," the second to "Prolegomena to Cardinal Arithmetic."

The authors introduce the concept of types which is intended to serve the purpose of avoiding contradictions. The two chief notations used therewith are (Vol. I, p. 419):

$t_0'\alpha$ meaning the type in which α is contained.

$t'x$ meaning the type of which x is a member.

"The type of members of α" is defined thus,

$t_0'\alpha = \alpha \cup - \alpha$ Df.

If $\iota'x$ is the class whose only member is x, we define $t'x = \iota'x \cup - \iota'x$ Df. It follows that $\vdash . t'x = t_0'\iota'x$.

"If R is any relation (Whitehead and Russell, Vol. I, p. 429) it is of the same type as $t_0'D'R \uparrow t_0'\text{Ɑ}'R$. If $D'R$ and $\text{Ɑ}'R$ are both of the same type as α, R is of the same type as $t_0'\alpha \uparrow t_0'\alpha$, which is of the same type as $\alpha \uparrow \alpha$. The type of $t_0'\alpha \uparrow t_0'\alpha$ we call $t_{00}'\alpha$, and the type of $t^{m}{}'\alpha \uparrow t^{n}{}'\alpha$ we call $t^{mn}{}'\alpha$, and the type of $t_m'\alpha \uparrow t_n'\alpha$ we call $t_{mn}'\alpha$, and the type of $t_m'\alpha \uparrow t^{n}{}'\alpha$ we call $t_m^{n}{}'\alpha$, and the type of $t^{m}{}'\alpha \uparrow t_n'\alpha$ we call ${}^{m}t_n'\alpha$. We thus have a means of expressing the type of any relation R in terms of the type of α, provided the types of the domain and converse domain of R are given relatively to α."

If α is an ambiguous symbol (*ibid.*, p. 434) representing a class (such as Λ or V, for example),

α_x denotes what α becomes when its members are determined as belonging to the type of x, while

$\alpha(x)$ denotes what α becomes when its members are determined as belonging to the type of $t'x$. Thus,

V_x is everything of the same type as x, i.e., $t'x$.

$V(x)$ is $t't'x$.

$\alpha \, sm \, \beta$ means "α is similar to β," i.e., there is a one-one relation whose domain is α and whose converse domain is β (*ibid.*, p. 476). We have

$$\alpha \; sm \; \beta \, . \equiv . \, (\exists R) \, . \, R \epsilon 1 \rightarrow 1 \, . \, \alpha = D'R \beta = \text{Ɑ}R \, .$$

$k \epsilon Cls^2 \, excl$ means that k is a class of mutually exclusive classes (*ibid.*, p. 540).

$k \epsilon Cls \, ex^2 \, excl$ means the foregoing when no member of k is null.

$Cl \, excl ' \gamma$ means a $Cls^2 \, excl$ which is contained in a class of classes γ.

R_{*} means an ancestral relation to a given relation R (*ibid.*, p. 569).

$x R_{*} y$ means that x is an ancestor of y with respect to R (p. 576).

xBP means $x \epsilon D'P - \text{Ɑ}'P$ (*ibid.*, p. 607).

$x \, min_p \, \alpha$ means $x \epsilon \alpha \frown C`P - \breve{P}``\alpha$,
i.e., x is a member of α and of $C`P$, and no member of $\alpha$ precedes $x$ in $C`P$.

$\overleftarrow{R}_{*}`x$ means the posterity of a term with respect to the relation R (*ibid.*, p. 637).

$(\overleftarrow{R}_{*}`x) \upharpoonleft R$ means the relation R confined to the posterity of x.

In the second volume of the *Principia mathematica* (1912) Whitehead and Russell treat of "Cardinal Arithmetic" (Part III), "Relation-Arithmetic" (Part IV), and begin the subject of "Series" (Part V), which they continue in the third volume (which was brought out in 1913). This third volume gives also a treatment of "Quantity" (Part VI).

Some new notations and fresh interpretations of symbols previously used are introduced in the last two volumes. In the second volume the theory of types is continued and applied to cardinal number. "Contradictions concerning the maximum cardinal are solved by this theory" (Vol. II, p. 3).

$Nc`\alpha$ means the class of all classes similar to α, i.e., as $\hat{\beta}(\beta \, sm \, \alpha)$. This definition is due to Frege and was first published in 1884. For formal purposes of definition, the authors put

$Nc = \overrightarrow{sm}$ Df.

$NC = D`Nc$ Df. (*ibid.*, p. 5).

$k \, \overline{sm} \, \overline{sm} \, \lambda = (1 \rightarrow 1) \frown \overleftarrow{\text{Cl}}`S`\lambda \frown \hat{T}(k = T_{\epsilon}``\lambda)$ Df. (*ibid.*, p. 88). If P and Q are generating relations of two series (*ibid.*, p. 347),

$P \ddagger Q = P \cup Q \cup C`P \uparrow C`Q$ Df.

Lack of space prevents us from enumerating all the specialized notations adopted in the treatment of particular topics.

696. *Signs of Poretsky.*—The Russian, Platon Poretsky,[1] uses a symbolism of his own. For example, he starts out with three known forms of the logical equality $A = B$, which he expresses in the form $(A = B) = (A_0 = B_0) = (AB_0 + A_0B = 0) = ((AB + A_0B_0 = 1)$ where A_0 and B_0 are the negations of the classes A and B. Letting N stand for *nihil* or the complete zero, M for *mundus* or the logical all, he rewrites the foregoing relations thus:

$$(A = B) = (A_0 = B_0) = (0 = N) = (1 = M) \; ,$$

where $N = M_0 = AB_0 + A_0B$, $M = N_0 = AB + A_0B_0$.

[1] P. Poretsky, *Sept Lois fondamentales de la théorie des égalités logiques* (Kazan, 1899), p. 1.

In logical inequalities[1] he takes $A<B$, meaning "A is contained in B" or $A=AB$; $A>B$, meaning "A contains B" or $A=A+C$; $A\mp B$, meaning "A is not equal to B." The author uses $\because$ for "since" and $\therefore$ for "therefore."

Signs of Julius König are found in his book[2] of 1914. For example, "a rel$_\alpha$ b" stands for the declaration that "the thing a" corresponds α-wise to "the thing b"; $\veebar$ is the sign for isology, $\check{\times}$ for conjunction, $\check{+}$ for disjunction, $(=)$ for equivalence.

697. *Signs of Wittgenstein.*—A partial following of the symbols of Peano, Whitehead, and Russell is seen in the *Tractatus Logico-Philosophicus* of Ludwig Wittgenstein. "*That a* stands[3] in a certain relation to b says *that aRb*"; $\sim p$ is "not p," $p \vee q$ is "p or q" (p. 60, 61). "If we want to express in logical symbolism the general proposition 'b is a successor of a' we need for this an expression for the general term of the formal series: aRb, $(\exists x):aRx.xRb$, $(\exists x, y):aRx.xRy.yRb$, ..." (p. 86, 87).

"When we conclude from $p \vee q$ and $\sim p$ to q the relation between the forms of the propositions '$p \vee q$' and '$\sim p$' is here concealed by the method of symbolizing. But if we write, e.g., instead of '$p \vee q$,' '$p \mid q \cdot\mid\cdot p \mid q$' and instead of '$\sim p$', '$p \mid p$' ($p \mid q$=neither p nor q, then the inner connexion becomes obvious" (p. 106, 107).

Our study of notations in mathematical logic reveals the continuance of marked divergence in the symbolism employed. We close with some general remarks relating to notations in this field.

698. *Remarks by Rignano and Jourdain.*—Jourdain says:[4] "He [Rignano][5] has rightly emphasized the exceedingly important part that analogy plays in facilitating the reasoning used in mathematics, and has pointed out that a great many important mathematical discoveries have been brought about by this property of the symbolism used. Further he held that the symbolism of mathematical logic is

[1] P. Poretsky, *Théorie des non-égalités logiques* (Kazan, 1904), p. 1, 22, 23.

[2] Julius König, *Neue Grundlagen der Logik, Arithmetik und Mengenlehre* (Leipzig, 1914), p. 35, 73, 75, 81.

[3] Ludwig Wittgenstein, *Tractatus Logico-Philosophicus* (English ed.; London, 1922), p. 46, 47.

[4] P. E. B. Jourdain, "The Function of Symbolism in Mathematical Logic," *Scientia*, Vol. XXI (1917), p. 1–12.

[5] Consult Eugenio Rignano in *Scientia*, Vol. XIII (1913), p. 45–69; Vol. XIV (1913), p. 67–89, 213–39; Vol. XVII (1915), p. 11–37, 164–80, 237–56. See also G. Peano, "Importanza dei simboli in matematica," *Scientia*, Vol. XVIII (1915), p. 165–73.

not to be expected to be so fruitful, and, in fact, that it has not been so fruitful.

"It is important to bear in mind that this study is avowedly concerned with the psychological aspect of symbolism, and consequently such a symbolism as that of Frege, which is of much less service in the economy of thought than in the attainment of the most scrupulous precision, is not considered. . . . .

"In Peano's important work, dating from 1888 onwards, on logical and mathematical symbolism, what he always aimed at, and largely succeeded in attaining, was the accurate formulation and deduction from explicit premises of a number of mathematical theories, such as the calculus of vectors, arithmetic, and metrical geometry. It was, then, consistently with this point of view that he maintained that Rignano's criticisms hold against those who consider mathematical logic as a science in itself, but not against those who consider it as an instrument for solving mathematical problems which resist the ordinary methods.

"The proper reply to Rignano seems, however, to be that, until comparatively lately, symbolism in mathematics and the algebra of logic had the sole aim of helping reasoning by giving a fairly thorough analysis of reasoning and a condensed form of the analyzed reasoning, which should, by suggesting to us analogies in familiar branches of algebra, make mechanical the process of following the thread of deduction; but that, on the other hand, a great part of what modern mathematical logic does is to increase our subtlety by emphasizing *differences* in concepts and reasonings instead of *analogies*."

699. *A question.*—No topic which we have discussed approaches closer to the problem of a uniform and universal language in mathematics than does the topic of symbolic logic. The problem of efficient and uniform notations is perhaps the most serious one facing the mathematical public. No group of workers has been more active in the endeavor to find a solution of that problem than those who have busied themselves with symbolic logic—Leibniz, Lambert, De Morgan, Boole, C. S. Peirce, Schröder, Peano, E. H. Moore, Whitehead, Russell. Excepting Leibniz, their mode of procedure has been in the main individualistic. Each proposed a list of symbols, with the hope, no doubt, that mathematicians in general would adopt them. That expectation has not been realized. What other mode of procedure is open for the attainment of the end which all desire?

III

SYMBOLS IN GEOMETRY (ADVANCED PART)

1. RECENT GEOMETRY OF TRIANGLE AND CIRCLE, ETC.

700. Lack of space prevents the enumeration of the great masses of symbols occurring in the extensive literature on the recent geometry of the triangle and circle. There exists an almost hopeless heterogeneity in symbolism. This is evident from a comparison of the publications of J. Lange,[1] A. Emmerich,[2] C. V. Durell,[3] and Godfrey and Siddons.[4] Lange represents a triangle by ABC; the feet of the altitudes by $A.$, $B.$, $C.$; the intersection of the altitudes by H; the center of the circumcenter by M; the centers of the circles of contact by J, J_a, J_b, J_c; the point of contact of J with BC by X; the intersection of AJ with BC by D; the diameter of the circumcircle that is perpendicular to BC by EF; the radius of the circumcenter by r; the midpoints of the sides by A', B', C'; the midpoints of the upper parts of the altitudes by A_0, B_0, C_0; the center of gravity by S; the center of the circumcircle of $A'B'C'$ by M'; the radius of the incircle by ρ.

Emmerich's designation of terms and list of signs are in part as follows: The Brocard points (p. 23) are Ω and Ω'; the Grebe point (p. 23) is K; the circumcenter of a triangle (p. 29) is O; the collineation center of the first Brocard triangle $\triangle$ (p. 85), D; the center of the Feuerbach circle (p. 97) is F; the intersection of a median (p. 118) from the vertex A of a triangle with the opposite side g_a; the angle that this median makes with the side b is $\angle(g_{a,b})$; the centers of the Neuberg circles (p. 131) are N_a, N_b, N_c; homologous points respective to the sides a, b, c of a $\triangle ABC$ (p. 138) are J_a, J_b, J_c; the centers of McKay circles (p. 142) are M_a, M_b, M_c; antiparallelism is denoted (p. 13) by $a||$.

[1] Julius Lange, *Geschichte des Feuerbachschen Kreises* (Berlin, 1894), p. 4.

[2] A. Emmerich, *Die Brocardschen Gebilde* (Berlin, 1891). See also the "Anhang."

[3] Clement V. Durell, *Modern Geometry. The Straight Line and Circle* (London, 1920).

[4] C. Godfrey and A. W. Siddons, *Modern Geometry* (Cambridge, 1908), p. 16.

The Ω and Ω' for Brocard points are found also in Lachlan's treatise.[1]

The symbols of Godfrey and Siddons, and Durell, are in partial agreement. Godfrey and Siddons give the following list of symbols:

A, B, C vertices of the triangle	$\triangle$ area of the triangle
D, E, F feet of the altitudes	S circumcenter
α, β, γ midpoints of the sides	H orthocenter
X, Y, Z points of contact of the incircle	G centroid
a, b, c lengths of the sides	I incenter
s semiperimeter	I_1, I_2, I_3 excenters
R circumradius	N nine-points center
r inradius	P, Q, R midpoints of HA, HB, HC
r_1, r_2, r_3 exradii	

701. *Geometrography.*—Lemoine[2] in his *Géométrographie* used:

C_1 to mark the placing a point of the compasses at a fixed point

C_2 to mark the placing a point of the compass at an undetermined point of a line

C_3 drawing the circle

R_1 placing a ruler at a point

R_2 drawing a straight line

He counts the number of steps in a geometric construction by the polynomial

$$l_1 \cdot R_1 + l_2 \cdot R_2 + l_3 \cdot C_1 + l_4 \cdot C_2 + l_5 \cdot C_3$$

and represents the degree of simplicity by the sum

$$l_1 + l_2 + l_3 + l_4 + l_5$$

and the degree of precision by

$$l_1 + l_3 + l_4 \,.$$

702. *Signs for polyhedra.*—In the expression of the relation between the edges, vertices, and faces of a convex polyhedron the initial letters of the words for "edges," "vertices," "faces" have usually been employed. Hence the designations have varied with the language used. The theorem was first enunciated by Descartes, but his

[1] R. Lachlan, *Modern Pure Geometry* (London, 1893), p. 65

[2] E. Lemoine, *Comptes rendus*, Vol. CVII (1888), p. 169. See also J. Tropfke, *op. cit.*, Vol. IV (2d ed., 1923), p. 91.

treatment of it was not published until 1859 and 1860.[1] It was found in a manuscript of Leibniz in the Royal Library of Hanover and is an excerpt made by Leibniz, during his residence in Paris in the years 1675–76, from a Cartesian manuscript. Descartes says: "Ponam semper pro numero angularum solidorum α et pro numero facierum φ," that is, α is the number of vertices (*anguli solidi*) and φ is the number of faces (*facies*). He states the theorem thus:[2] "Numerus verorum angulorum planorum est $2\varphi+2\alpha-4$." ("The number of polygonal angles is $2\varphi+2\alpha-4$.") As the sum of the polygonal angles is twice as great as the number of edges, it is evident that Descartes' phrasing is equivalent to the statement: faces +vertices = edges+2. Euler,[3] after whom the theorem is named, gave it the form $S+H=A+2$, where S = *numerus angulorum solidorum* (vertices), A = *numerus acierum* (edges), H = *numerus hedrarum* (faces). Euler's letters were retained by Legendre,[4] Poinsot,[5] and Cayley,[6] the last two in their discussions introducing some half-dozen additional letters.

In the statement of the theorem, German writers[7] usually employ the letters e (*Ecken*), k (*Kanten*), f (*Flächen*); English writers,[8] e ("edges"), v ("vertices"), f ("faces").

In the morphology of polyhedra unusual symbols appear here and there. V. Eberhard[9] explains the notation used by him thus:

According as the plane α_n contains a three-, four-, or five-sided face of the n-faced A_n, it cuts from the $(n-1)$-faced A_{n-1} a three-edged vertex $(\alpha_i, \alpha_k, \alpha_l)$, or an edge $|\alpha_k, \alpha_l|$, or two edges meeting in a vertex $|\alpha_l\ \alpha_k,|$ and $|\alpha_i, \alpha_l|$. For these three operations Eberhard selects, respectively, the symbolism

$$\alpha_n \dot{-} (\alpha_i, \alpha_k, \alpha_l), \qquad \alpha_n \underline{\overset{\blacksquare}{}} |\alpha_k, \alpha_l|, \qquad \alpha_n \underline{\triangle} [\,|\alpha_i, \alpha_k|, |\alpha_i, \alpha_l|\,] .$$

[1] *Œuvres de Descartes* (éd. C. Adam et P. Tannery), Vol. X (Paris, 1908), p. 257.

[2] *Op. cit.*, Vol. X, p. 269.

[3] L. Euler, *Novi Comment. Acad. Petropol. ad annum 1752 et 1753* (1758), Vol. IV, p. 119, 158–60. See also M. Cantor, *Vorlesungen*, Vol. III (2. Aufl.), p. 557.

[4] A. M. Legendre, *Éléments de géométrie* (Paris, 1794), p. 228.

[5] L. Poinsot, *Jour. Polyt.*, Vol. IV (1810), p. 16–48.

[6] A. Cayley, *London, Edinburgh and Dublin Philosophical Magazine* (4th ser.), Vol. XVII (1859), p. 123–28.

[7] See, for instance, H. Durège, *Elemente der Theorie der Funktionen* (Leipzig, 1882), p. 226

[8] For instance, W. W. Beman and D. E. Smith, *New Plane and Solid Geometry* (Boston, 1899), p. 284.

[9] V. Eberhard, *Zur Morphologie der Polyeder* (Leipzig, 1891), p. 17, 29.

If, in a special case, the plane a_n passes through several vertices, $a, b, \ldots .$ of A_{n-1}, this is expressed by

$$(a, b, \ldots .)\ a_n \overset{\cdot}{\overline{}},\ \overset{\blacksquare}{\overline{}},\ \underline{\wedge}\ .$$

If a_i is a plane (p. 29) of the polyhedron A_n, then the m-sided polygon bounding the face is designated by $\langle a_i \rangle_m$. Two faces, $\langle a_i \rangle$ and $\langle a_k \rangle$, can be in direct connection with each other in three ways: They have one edge in common, for which we write $\langle a_i \rangle | \langle a_k \rangle$; or there exist outside of the two faces m edges such that each edge connects a vertex of the one face with a vertex of the other face, for which we

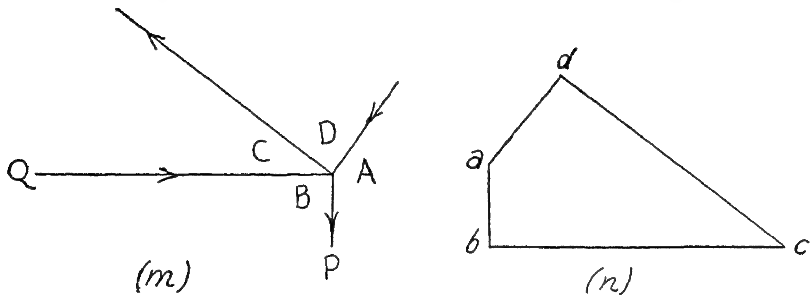

FIG. 126.—Bow's notation

write $\langle a_1 \rangle\rangle \overline{m} \langle\langle a_k \rangle$; or both of these circumstances occur simultaneously, for which we write $\langle a_i \rangle\, |, \rangle \underset{m}{\overline{}} \langle\langle a_k \rangle$.

Max Brückner[1] designates the relation between three planes (p), (q), (r) without common edges by the symbols $(p) \underset{x}{\overline{}} (q)$, $(q) \underset{y}{\overline{}} (r)$, $(r) \underset{z}{\overline{}} (p)$, where x, y, z are the numbers of edges (*Scheitelkanten*) to be determined.

703. *Geometry of graphics.*—The civil engineer of Edinburgh, R. H. Bow,[2] devised the notation shown in Figure 126, known as "Bow's notation," in practical graphics. Diagram (m) shows the lines of action of a number of forces which are in equilibrium. Instead of denoting a force by a single letter, P, as is practiced in some systems, the force is here designated by two letters which are placed on opposite sides of the line of action of the force in the fram-diagram (m), and at the angular points of the polygon in force-diagram (n). Thus the force P is referred to in Bow's notation as AB.

[1] Max Brückner, *Vielecke und Vielflache. Theorie and Geschichte* (Leipzig, 1900), p. 87.

[2] Robert Henry Bow, *Economics of Construction* (London, 1873), p. 53–55. Our figure is taken from D. A. Low, *Practical Geometry and Graphics* (London, 1912), p. 92.

The angle of contact[1] of a curve $y=ax^m$ (where $a>0$ and $m>1$) with the axis of X as its tangent is designated by Giulio Vivanti by the symbol (a, m), so that $(a, m)>(a', m)$ when $a>a'$, and $(a, m)<(a', m')$ when $m>m'$.

2. PROJECTIVE AND ANALYTICAL GEOMETRY

704. Poncelet,[2] in his *Traité des propriétés projectives des figures*, uses little symbolism. He indicates points by capital letters A, B, C, ... , and the center of projection by S.

705. *Signs for projectivity and perspectivity.*—Von Staudt, in his *Geometry of Position* (1847), expressed himself as follows:[3] "Gleichwie ein Gebilde, welches aus den Elementen A, B, C, D besteht, gewöhnlich durch $ABCD$ bezeichnet wird, so soll ein Gebilde, welches aus den Elementen SA, SB, SC, SD besteht, durch $S(ABCD)$ bezeichnet werden." Again:[4] "Zwei einförmige Grundgebilde heissen zu einander projektivisch ($\wedge$), wenn sie auf einander bezogen sind, dass jedem harmonischen Gebilde in dem einen ein harmonisches Gebilde im andern entspricht." Also:[5] "Wenn in zwei einander projektivischen Grundgebilden je zwei homologe Elemente P, P_1 einander doppelt (abwechselnd) entsprechen, nämlich dem Punkte P des einen Gebildes das Element P_1 des anderen, und dem Elemente P_1 des ersteren das Element P des letzteren entspricht, so heissen die Gebilde involutorisch liegend ($\doublebarwedge$)."

Here we have introduced the two noted signs $\barwedge$ and $\doublebarwedge$. Neither came to be adopted by all prominent writers on projective geometry, but $\barwedge$ has met with far wider acceptance than the other. One finds $\barwedge$ employed by Hankel,[6] Reye,[7] Fiedler,[8] Halsted,[9] Aschieri,[10] Sturm,[11]

[1] C. Isenkrahe, *Das Endliche und das Unendliche* (Münster, 1915), p. 306.

[2] J. V. Poncelet, *Traité des propriétés projectives des figures* (2d ed.), Vol. I (Paris, 1865), p. 6. (First edition appeared in 1822.)

[3] Georg Kane Christian Von Staudt, *Geometrie der Lage* (Nürnberg, 1847), p. 32.

[4] *Op. cit.*, p. 49.

[5] *Op. cit.*, p. 118. Von Staudt uses this symbolism also in his *Beiträge*, fasc. 3, p. 332.

[6] H. Hankel, *Elemente der projectivischen Geometrie* (Leipzig, 1875), p. 79.

[7] Theodor Reye, *Die Geometrie der Lage*, Vol. I (2d ed.; Leipzig, 1882), p. 121. (First edition, 1866.)

[8] W. Fiedler, *Darstellende Geometrie*, 1. Theil (Leipzig, 1883), p. 31.

[9] G. B. Halsted, *Elementary Synthetic Geometry* (New York, 1892), p. vii.

[10] Ferdinando Aschieri, *Geometria projettiva* (Milano, 1895), p. 162.

[11] Rudolf Sturm, *Liniengeometrie* (Leipzig, 1896), p. 8.

Henrici and Treutlein,[1] Bobek,[2] Böger,[3] Enriques,[4] Amodeo,[5] Doehlemann,[6] Filon,[7] Schur,[8] Guareschi,[9] Schoenflies and Tresse,[10] Holgate,[11] and Dowling.[12] Frequently the notation $\overline{\wedge}$ is used in modern geometry to indicate a homology of points; thus, $ABC \ldots . MN \ldots . \overline{\wedge} A'B'C' \ldots . M'N'$ indicates that $ABC \ldots . MN \ldots .$ are homologues with $A'B'C' \ldots . M'N' \ldots .$ in two homographic figures of range 1, 2, or 3.[13] This same notation is used also to designate homologous elements of two planes or two sheaves, or two spaces, in homographic correspondence.

The sign $\overline{\overline{\wedge}}$ to mark perspective position is found in Böger,[14] Doehlemann,[15] Holgate,[16] and Dowling. Some prominent writers, Cremona[17] for instance, make no use either of $\overline{\overline{\wedge}}$ or $\overline{\wedge}$.

706. *Signs for harmonic and anharmonic ratios.*—Möbius[18] in 1827 marked the anharmonic ratio (*Doppelschnittsverhältniss*) of four

[1] J. Henrici and P. Treutlein, *Lehrbuch der Elementar-Geometrie*, 2. Teil (Leipzig, 1897), p. 10, 42.

[2] Karl Bobek, *Einleitung in die projektivische Geometrie der Ebene* (Leipzig, 1897), p. 7.

[3] R. Böger, *Elemente der Geometrie der Lage* (Leipzig, 1900), p. 17.

[4] Federigo Enriques, *Vorlesungen über Projektive Geometrie*, deutsch von H. Fleischer (Leipzig, 1903), p. 106.

[5] F. Amodeo, *Lezioni di geometria proiettiva* (Napoli, 1905), p. 60.

[6] Karl Doehlemann, *Projektive Geometrie*, 3. Aufl. (Leipzig, 1905), p. 78.

[7] L. N. G. Filon, *Introduction to Projective Geometry* (London, [1908]), p. 46.

[8] F. Schur, *Grundlagen der Geometrie* (Leipzig und Berlin, 1909), p. 60.

[9] G. Guareschi in E. Pascal's *Repertorium* (2d ed.), Vol. II_1 (Leipzig, 1910), p. 114.

[10] A. Schoenflies and A. Tresse in *Encyclopédie des scien. math.*, Tome III, Vol. II (1913), p. 32, 49.

[11] T. F. Holgate in J. W. A. Young's *Monographs on .... Modern Mathematics* (New York, 1911), p. 75.

[12] L. W. Dowling, *Projective Geometry* (New York, 1917), p. 19, 38.

[13] A. Schoenflies and A. Tresse in *Encyclopédie des scien. math.*, Tom. III, Vol. II (1913), p. 32.

[14] Rudolf Böger, *Elemente der Geometrie der Lage* (Leipzig, 1900), p. 14.

[15] Karl Doehlemann, *Projektive Geometrie*, 3. Aufl. (Leipzig, 1905), p. 132.

[16] T. F. Holgate, *op. cit.*, p. 75.

[17] Luigi Cremona, *Projective Geometry* (trans. Ch. Leudesdorf; 2d ed.; Oxford, 1893).

[18] A. F. Möbius, *Der barycentrische Calcul* (Leipzig, 1827); *Gesammelte Werke*, Vol. I (Leipzig, 1885), p. 221.

collinear points, that is, the ratio of ratios, $\frac{AC}{CB}:\frac{AD}{DB}$, in which AB is divided by C and D, by the symbolism (A,B,C,D).

Chasles[1] in 1852 did not use any special notation and denoted the anharmonic ratio, in which cd is divided by a and b, by $\frac{ac}{ad}:\frac{bc}{bd}$. Salmon[2] marks the anharmonic ratio of four points A, B, C, D by $\frac{AB}{BC}\div\frac{AD}{DC}$ and uses the abbreviation $\{ABCD\}$, and, for the anharmonic ratio of pencils, $\{O.ABCD\}$. Salmon omits the commas. Clifford[3] writes this relation for the points A,B,D,E in the form $[ADBE]$. In 1865 Chasles[4] adopts for the anharmonic ratio of four points a, b, c, d the notation (a, b, c, d), and for pencils (O_a, O_b, O_c, O_d) or $O(a, b, c, d)$. J. W. Russell[5] writes the anharmonic ratio $\frac{BC}{CA}\div\frac{BD}{DA}$ of four collinear points A, B, C, D in the form (BA, CD); and, correspondingly, $V(BA, CD)$. Others prefer braces,[6] as in $\{AB, PQ\}$. W. A. Whitworth[7] uses $\{p.abcd\}$ to denote the anharmonic ratio of the points in which the straight lines a, b, c, d intersect the straight line p.

707. *Signs in descriptive geometry.*—In this field of geometry gross diversity of notation has existed. Monge[8] had a simple symbolism. The projection of a point upon the horizontal plane he marked by a capital letter, say D, and the projection on the vertical plane by the corresponding small letter d; the projections of a line were marked AB and ab, respectively. The traces of a plane he represents by capital letters, such as AB, BC. He uses single and double accents for the marking of certain corresponding points.

Aubré[9] uses the letters h ("horizontal") and v ("vertical"), and

[1] M. Chasles, *Traité de géométrie supérieure* (Paris, 1852), p. 28, 29, 183, etc.

[2] George Salmon, *Conic Sections* (6th ed.; London, 1879), § 326, p. 297, 301; the first edition appeared in 1848; the third in 1855.

[3] W. K. Clifford in *Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. II (1864), p. 233; *Mathematical Papers* (London, 1882), p. 27.

[4] M. Chasles, *Traité des sections coniques* (Paris, 1865), p. 7, 8.

[5] J. W. Russell, *Pure Geometry* (Oxford, 1893), p. 98, 102.

[6] R. Lachlan, *Modern Pure Geometry* (London, 1893), p. 266.

[7] William Allen Whitworth, *Trilinear Coordinates* (Cambridge, 1866), p. 381.

[8] *Géométrie descriptive*, par G. Monge (5th ed., M. Brisson; Paris, 1827), p. 19, 20. The first edition bears the date of An III (1795).

[9] *Cours de géométrie descriptive, d'après la méthode de M. Th. Olivier*, par L. E. Aubré (1856), p. 6. Oliver's first edition (1843) and second (1852) contained the notations described by Aubré.

designates the projection upon the horizontal and vertical planes of a point o by o^h and o^v. Aubré marks the four right dihedral angles formed by the interesting horizontal and vertical planes by $\widehat{A,S}$, $\widehat{P,S}$, $\widehat{P,I}$, $\widehat{A,I}$, where A stands for *antérieur*, S for *supérieur*, P for *postérieur*, I for *inférieur*. A straight line in space is marked by capital letter, say D, and its projections by D^h and D^v. Church[1] denoted a point in space by, say, M; its horizontal projection by m; and its vertical by m'; a line by, say, MN. Wiener[2] represents points by Latin capitals, surfaces by black-faced Latin capitals, lines by small Latin letters, angles by small Greek letters. Berthold[3] marks a point by, say, a; its horizontal projection by a'; and its vertical by a''. He marks a line by, say, ab. Bernhard[4] prefers small Latin letters for points, Latin capitals for straight lines and surfaces, small Greek letters for angles. He uses $\times$ for intersection, so that $a=G\times L$ means that the point a is the intersection of the coplanar lines G and L; P_1 and P_2 are the two mutually perpendicular planes of projection; X the axis of projection; a' and a'' are the projections of the point a, G' and G'' of the line G; g_1 and g_2 are the traces of G; E_1 and E_2 are the traces of the plane E.

Different again is the procedure of Blessing and Darling.[5] With them, planes are represented by capital letters, the last letters of the alphabet being used, thus, P, Q, R, S, T, U; the horizontal trace of a plane has the letter H prefixed, as in HP, HQ, etc.; the vertical trace has the letter V prefixed, as in VP, VQ, etc. The ground line is marked $G-L$; a line in space $C-D$ has two projections, $c-d$ and $c'-d'$.

The illustrations cited suffice to show the lack of uniformity in notation.[6] They indicate, moreover, that the later books on descriptive geometry introduce a larger amount of symbolism than did the earlier.

[1] Albert Church, *Elements of Descriptive Geometry* (New York, 1870), p. 2, 5.

[2] Christian Wiener, *Lehrbuch der Darstellenden Geometrie*, Vol. I (Leipzig, 1884), p. 62.

[3] *Die Darstellende Geometrie*, von W. H. Behse, bearbeitet von P. Berthold, 1. Theil (Leipzig, 1895), p. 3, 11.

[4] Max Bernhard, *Darstellende Geometrie* (Stuttgart, 1909), p. 3.

[5] George F. Blessing and Lewis A. Darling, *Elements of Descriptive Geometry* (New York, 1913), p. 19, 20, 49.

[6] Consult also G. Loria, "I doveri dei simboli matematici," *Bollettino della Mathesis* (Pavia, 1916), p. 35.

708. *Signs in analytical geometry.*—Descartes introduced the letters $z, y, x, \ldots$ as variable co-ordinates (§§ 339, 340). With him a letter represented as a rule only a positive number. It was Johann Hudde who in 1659 first let a letter stand for negative, as well as positive, values (§ 392). In Descartes' *Géométrie* (1637), the co-ordinate axes are in no case explicitly set forth and drawn. He selects a straight line which he sometimes calls a "diameter" and which serves as the x-axis. The formal introduction of the y-axis is found in G. Cramer,[1] though occasional references to a y-axis are found earlier in De Gua, L. Euler, W. Murdock, and others. With Descartes an equation of a geometric *locus* is not considered valid except for the angle of the co-ordinates (quadrant) in which it was established. The extension of an equation to other quadrants was freely made in particular cases for the interpretation of negative roots.

Polar co-ordinates were introduced in a general manner by Jakob Bernoulli[2] and Pierre Varignon.[3] The latter puts $x=\rho$ and $y=l\omega$ and gets wholly different curves; the parabolas $x^m=a^{m-1}y$ become the Fermatian spirals.

An abridged notation and new systems of co-ordinates were introduced by E. Bobillier[4] and by Julius Plücker. Bobillier writes the equations of the sides of a triangle in the form $A=0$, $B=0$, $C=0$, and the conic passing through the three vertices of the triangle, $aBC+bCA+cAB=0$, where a, b, c may take any values. Bobillier proves Pascal's theorem on a hexagon inscribed in a conic in this manner:[5] Let $aAA'-bBB'=0$ be the conic circumscribing a quadrilateral, then the six sides of the inscribed hexagon may be written $A=0$, $\gamma A+bB=0$, $\gamma B'+aA'=0$, $A'=0$, $aA'+\gamma' B=0$, $\gamma' A+bB'=0$. The Pascal line is then $\gamma\gamma' A-abA'=0$. Plücker used abridged notations and formulated the principles of duality and homogeneity. The abridged notations are used extensively in several chapters of Salmon's *Conic Sections* (London, 1848), and in later editions of that work.

[1] G. Cramer, *Introduction à l'analyse des lignes courbes algébriques* (1750).

[2] Jakob Bernoulli, *Opera*, Vol. I (Geneva, 1744), p. 578–79. See also G. Eneström, *Bibliotheca mathematica* (3d ser.), Vol. XIII (1912–13), p. 76.

[3] *Mémoires de l'académie r. d. sciences*, année 1704 (Paris, 1722).

[4] E. Bobillier in Gergonne's *Annales de mathématiques*, Vol. XVIII (1828), p. 320, etc. See also Julius Plücker, *Gesammelte Mathematische Abhandlungen*, Vol. I (Leipzig, 1895), p. 599.

[5] E. Bobillier in Gergonne's *Annales*, Vol. XVIII, p. 359–67. See also E. Kötter, *Entwickelung der synthetischen Geometrie* (Leipzig, 1901), p. 27 n.

709. *Plücker's equations* on the ordinary singularities of curves exhibit considerable variety, among different authors, in the use of letters. Thus, the order (degree) of a curve is marked n by Plücker,[1] Berzolari,[2] and Kötter,[3] and m by Salmon;[4] the class of the curve is marked m by Plücker and Kötter, n' by Berzolari, n by Salmon; the number of double points is marked x by Plücker, δ by Salmon and Kötter, d by Berzolari; the number of double tangents is marked u by Plücker, τ by Salmon and Kötter, d' by Berzolari; the number of stationary points (*Rückkehrpunkte, Spitzen*) is marked y by Plücker, κ by Kötter and Salmon, r by Berzolari; the number of stationary tangents (*Wendepunkte, Wendetangenten*) is marked v by Plücker, ι by Salmon and Kötter, r' by Berzolari.

710. *The twenty-seven lines on cubic surface.*—For the treatment of these lines, noteworthy notations have been invented. We give that of Andrew S. Hart, as described by Salmon:[5] "The right lines are denoted by letters of three alphabets, as follows:

$$
\begin{array}{lll}
A_1,\ B_1,\ C_1\ ; & A_2,\ B_2,\ C_2\ ; & A_3,\ B_3,\ C_3\ ; \\
a_1,\ b_1,\ c_1\ ; & a_2,\ b_2,\ c_2\ ; & a_3,\ b_3,\ c_3\ ; \\
\alpha_1,\ \beta_1,\ \gamma_1\ ; & \alpha_2,\ \beta_2,\ \gamma_2\ ; & \alpha_3,\ \beta_3,\ \gamma_3\ .
\end{array}
$$

Letters of the same alphabet denote lines which meet if either the letters or the suffixes be the same; for example, $A_1\ A_2\ A_3$ denote lines on the same plane as also do $A_1\ B_1\ C_1$. Letters of different alphabets denote lines which meet," according to a given table. Salmon introduced a notation of his own.

The notation most generally adopted is that of the "double-six," due to L. Schläfli,[6] of Bern. He represents a double six by

$$
\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6 \end{pmatrix}
$$

[1] Julius Plücker, *Theorie der algebraischen Curven* (Bonn, 1839), p. 211.

[2] Berzolari in Pascal's *Repertorium der höheren Mathematik* (ed. P. Epstein und H. E. Timerding), Vol. II (2d ed.; Leipzig und Berlin, 1910), p. 286.

[3] Ernst Kötter, *Entwickelung der synthetischen Geometrie* (Leipzig, 1901), p. 465.

[4] George Salmon, *Higher Plane Curves* (3d ed.; Dublin, 1879), p. 64.

[5] G. Salmon, *Cambridge und Dublin Mathematical Journal*, Vol. IV (1849), p. 252, 253. See also A. Henderson, *The Twenty-seven Lines upon the Cubic Surface* (Cambridge, 1911).

[6] L. Schläfli, *Quarterly Journal of Mathematics*, Vol. II (London, 1858), p. 116.

and finds that "no two lines of the same horizontal row and no two lines of the same vertical row intersect, but any two lines otherwise selected do intersect."

711. *The Pascal hexagram.*—The treatment of this complicated figure exhibiting its properties has given rise to various notations. With Plücker,[1] if (a) and (b) represented two points, then (a, b) represented the line joining them, and $[(a, b), (c, d)]$ or $[a, b; c, d]$ represented the intersection of the right lines (a, b), (c, d). A corresponding interpretation was given to these symbols when (a) and (b) represented two right lines. Plücker designated the vertices of a hexagon inscribed in a conic by the numerals (1), (2), (3), (4), (5), (6), and the line joining, say, (1) and (2), by (1, 2). Also, by $\left\{\begin{matrix}(1,2) & (2,3) & (3,4)\\ (4,5) & (5,6) & (6,1)\end{matrix}\right.$ he designated the right line passing through the three points in which the pairs of lines (1, 2) and (4, 5), (2, 3) and (5, 6), (3, 4) and (6, 1), intersect. He distinguishes the sixty different hexagons obtained by changing the sequence of the points (1), (2), (3), (4), (5), (6), by the order of the numerals, and arranges these hexagons into twenty groups, so that the sixth group, for example, is written

$$\left.\begin{matrix}1\ 2\ 3\ 4\ 5\ 6\\ 1\ 6\ 3\ 2\ 5\ 4\\ 1\ 4\ 3\ 6\ 5\ 2\end{matrix}\right\}\ \text{VI}$$

Upon the Pascal line of 1 2 3 4 5 6 lie the three points (1, 2; 4, 5), (2, 3; 5, 6), (3, 4; 6, 1). He found it convenient to write a group, say Group VI, also in the form

$$\begin{pmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{pmatrix}, \quad \begin{pmatrix}3 & 5 & 1\\ 2 & 4 & 6\end{pmatrix}, \quad \begin{pmatrix}1 & 5 & 3\\ 4 & 2 & 6\end{pmatrix}.$$

Kirkman[2] takes the numerals 1 2 3 4 5 6 to represent the vertices of the hexagon inscribed in a conic C and lets $12=0$ be the equation of the right line joining 1 and 2; $1\ 2 . 3\ 4 . 5\ 6 - \lambda 2\ 3 . 4\ 5 . 6\ 1 = 0 = CA$ is a Pascal line (A). The twenty points through which the sixty Pascal lines pass, three by three, are marked G. He denotes the line A by A 1 2 3 4 5 6, and the point G by G 1 2. 3 4 . 5 6. Pascal lines pass in addition, three together, through sixty points H, marked according to the plan H (12. 34) 56; the twenty points G lie, four together, on

[1] Plücker, *Crelle's Journal*, Vol. V (1830), p. 270–77.

[2] T. P. Kirkman, *Cambridge and Dublin Mathematical Journal*, Vol. V (1850), p. 185.

fifteen lines I, marked according to the plan I 12. 34. 56. The ninety lines, each containing two of the sixty points H, are marked J, according to the plan J 12. 43. The sixty points H lie three together on twenty lines, X.

Different notations are employed by von Staudt,[1] Cayley,[2] and Salmon.[3] A new notation was offered by Christine Ladd (Mrs. Franklin),[4] in which the vertices of the conic S are represented by A, B, C, D, E, F; the lines tangent to S at these vertices by a, b, c, d, e, f; the intersection of two fundamental lines AB, DE is called $P(AB.\ DE)$; the line joining two fundamental points ab, de is called $p'(ab.\ de)$. Here $p'(ab.de)$ is the pole of $P(AB.DE)$. The Pascal line of $ABCDEF$ is called $h(ABCDEF)$; it passes through the points $P(AB.DE)$, $P(BC.EF)$, $P(CD.FA)$. Similarly, the intersection of the lines $p'(ab.de)$, $p'(bc.ef)$, $p'(cd.fa)$ is the Brianchon point $H'(abcdef)$ of the hexagon $abcdef$. The three Pascal lines, $h(ABCFED)$, $h(AFCDEB)$, $h(ADCBEF)$, meet in a Steiner point $G(ACE.BFD)$. The four G points, $G(BDA.ECF)$ $G(EDF.BCA)$, $G(BCF.EDA)$, $G(BDF.ECA)$, lie on a Steiner-Plücker line $i(BE.CD.AF)$. The Kirkman point $H(ABCDEF)$ is the intersection of the Pascal lines $h(ACEBFD)$, $h(CEADBF)$, $h(EACFDB)$. A Cayley-Salmon line is marked $g(ACE.BFD)$; a Salmon point by $I(BE.CD.AF)$. Cayley "gives a table to show in what kind of a point each Pascal line meets every one of the 59 other Pascal lines. By attending to the notation of Pascal lines such a table may be dispensed with. His 90 points m, 360 points r, 360 points t, 360 points z, and 90 points w are the intersections each of two Pascals whose symbols can easily be derived one from another." For instance,

$$\left.\begin{matrix} h(ABCDEF) \\ h(ABC\ FED) \end{matrix}\right\rangle m\,.$$

"By the notation here given," continues the author, "it is immediately evident what points are on every line and what lines pass through every point, without referring to tables, as Veronese[5] is obliged to do."

[1] Von Staudt, *Crelle's Journal*, Vol. LXII (1863), p. 142–50.

[2] A. Cayley, *Quarterly Journal*, Vol. IX (1868), p. 268–74.

[3] G. Salmon, *Conic Sections* (6th ed., 1879), p. 379–83 n.

[4] C. Ladd, *American Journal of Mathematics*, Vol. II (1879), p. 1–12.

[5] Veronese's notation and additions to the Pascal hexagram are given in *Atti della R. Accademia dei Lincei* (3d ser.), "Memorie della classe de scienze fisiche ecc.," Vol. I (Rome, 1879), p. 649–703.

IV

THE TEACHINGS OF HISTORY

A. THE TEACHINGS OF HISTORY AS INTERPRETED BY VARIOUS WRITERS; INDIVIDUAL JUDGMENTS

712. *Review of André.*—A reviewer of Désiré André's book, *Notations mathématiques* (Paris, 1909), directs attention to the fact that, before the appearance of this book, no work existed that was devoted exclusively to mathematical notations. The reviewer says:[1] "They have at all times and have now more and more a very considerable influence upon the progress of the science. One may say that without them modern mathematics could not exist. They constitute therefore, in the study of the sciences, a subject of capital importance. One is therefore amazed that there exists no work consecrated entirely to them."

713. *Augustus de Morgan,* a close student of the history of mathematics, expresses himself, in part, as follows:[2] ". . . . Mathematical notation, like language, has grown up without much looking to, at the dictates of convenience and with the sanction of the majority. Resemblance, real or fancied, has been the first guide, and analogy has succeeded."

"Signs are of two kinds,—1st. Those which spring up and are found in existence, but cannot be traced to their origin; 2ndly, Those of which we know either the origin, or the epoch of introduction, or both. Those of the first kind pass into the second as inquiry advances. . . . .

"Mathematical marks or signs differ from those of written language in being almost entirely of the purely abbreviative character.

". . . . Too much abbreviation may create confusion and doubt as to the meaning; too little may give the meaning with certainty, but not with more certainty than might have been more easily attained. . . . .

"With the exception of an article by Mr. Babbage in the Edinburgh Encyclopædia, we do not know of anything written in modern times on notation in general.

[1] *Bulletin des sciences mathématiques* (2d ser.), Vol. XXXIII (October, 1909), p. 228.

[2] A. de Morgan, *Penny Cyclopaedia* (1842), art. "Symbols."

"We hardly need mention a thing so well known to the mathematician as that the progress of his science now depends more than at any previous time upon protection of established notation, when good, and the introduction of nothing which is of an opposite character. . . . . The language of the exact sciences is in a continual state of wholesome fermentation, which throws up and rejects all that is incongruous, obstructive and even useless. . . . .

"RULES.—Distinctions must be such only as are necessary, and they must be sufficient. . . . . The simplicity of notative distinctions must bear some proportion to that of the real differences they are meant to represent. . . . . Pictorial or descriptive notation is preferable to any other, when it can be obtained by simple symbols. . . . . Legitimate associations which have become permanent must not be destroyed, even to gain an advantage. . . . .

"A few Cambridge writers have of late years chosen to make a purely arbitrary change, and to signify by dy, dz, etc., not increments, but limiting ratios of increments: and students trained in these works must learn a new language before they can read Euler, Lagrange, Laplace, and a host of others. Thus $d_x y$ has been made to stand for $dy:dx$, and the old association connected with dy has (in the works spoken of) been destroyed. . . . .

"Notation may be modified for mere work in a manner which cannot be admitted in the expression of results which are to be reflected upon. The mathematical inquirer must learn to substitute, for his own private and momentary use, abbreviations which could not be tolerated in the final expression of results. . . . .

"Among the worst of barbarisms is that of introducing symbols which are quite new in mathematical, but perfectly understood in common, language. Writers have borrowed from the Germans the abbreviation $n!$ to signify $1.2.3 \ldots . (n-1)n$, which gives their pages the appearance of expressing surprise and admiration that 2, 3, 4, etc., should be found in mathematical results. . . . .

"The subject of mathematical printing has never been methodically treated, and many details are left to the compositor which should be attended to by the mathematician. Until some mathematician shall turn printer, or some printer mathematician, it is hardly to be hoped that this subject will be properly treated."

714. *J. W. L. Glaisher*[1] emphasizes the important rôle which notations have played in the development of mathematics: "Nothing

[1] J. W. L. Glaisher, "Logarithms and Computation" from the *Napier Tercentenary Memorial Volume* (ed. Cargill Gilston Knott; London, 1915), p. 75.

in the history of mathematics is to me so surprising or impressive as the power it has gained by its notation or language. No one could have imagined that such 'trumpery tricks of abbreviation' as writing $+$ and $-$ for 'added to' and 'diminished by,' or x^2, x^3, ... for xx, xxx, ... , etc., could have led to the creation of a language so powerful that it has actually itself become an instrument of research which can point the way to future progress. Without suitable notation it would be impossible to express differential equations, or even to conceive of them if complicated, much less to deal with them; and even comparatively simple algebraic quantities could not be treated in combination. Mathematics as it has advanced has constructed its own language to meet its need, and the ability of a mathematician in devising or extending a new calculus is displayed almost as much in finding the true means of representing his results as in the discovery of the results themselves.

"When mathematical notation has reached a point where the product of n x'es was replaced by x^n, and the extension of the law $x^m x^n = x^{m+n}$ had suggested $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x$ so that $x^{\frac{1}{2}}$ could be taken to denote $\sqrt{x}$, then fractional exponents would follow as a matter of course, and the tabulation of x in the equation $10^x = y$ for integral values of y might naturally suggest itself as a means of performing multiplication by addition. But in Napier's time, when there was practically no notation, his discovery or invention [of logarithms] was accomplished by mind alone without any aid from symbols."

715. *D. E. Smith*[1] directs attention to the teachings of history as aids to modern efforts to improve our notations: "In considering the improvement of our present algebraic symbolism, the possible elimination of certain signs that may have outlived their usefulness, the conservation of the best that have come down to us, the variation observed in passing from the literature of one country to that of another, and the possible unifying of the best of our present symbols, it has been found helpful to consider the early history of the subject."

716. *Alexandre Savérien* in the eighteenth century uttered a protest to the conditions then existing:[2] "Of what advantage is this multiplicity of characters? Is this expression for a square $\overline{a+b}^2$ not sufficiently simple and natural without recourse to this one ⊙ $\overline{a+b}^2$, and

[1] David Eugene Smith, in an Introductory Note to Suzan R. Benedict's "Development of Algebraic Symbolism from Paciuolo to Newton" in *Teachers College, Columbia University, Department of Mathematics* (1906–7), p. 11.

[2] Savérien *Dictionnaire universel de mathematique et de physique,* Vol. I (Paris, 1753), art. "Caractere."

is $\sqrt{aa+2ab+bb}$ not better than u which some wish to substitute for it? The reader will pardon me for writing thus. I could not let this occasion pass without stating what I think on this subject. Nothing is more pernicious than diversity in expressions. To invent new characters which signify nothing else than those which have been already accepted, is a wanton embroilment of things. If only one could once reach agreement on symbols, accord would continue. For a long time we have known that 6 signifies 'six'; what would we say, if some one instructed us to make it equal to 'seven'! Oh! What can be more useless, and more capable of disgusting a beginner and of embarrassing a geometer, than the three expressions ., :, ÷ to mark division? How could we guess that this meant to divide, rather than to multiply or to indicate a continued arithmetical progression, since the first and the last characters are designed, one for multiplication and one for progression? You may say all you wish about this, but I believe that the less one uses characters, the more one learns of mathematics. The memory is less burdened and consequently mathematical propositions are more easily mastered."

717. *Colin Maclaurin.*—The obscurity of concepts which has sometimes hidden behind mathematical symbols is treated by the eighteenth-century writer, Maclaurin, in the following passage:[1]

"The improvements that have been made by it [the doctrine of fluxions] . . . . are in a great measure owing to the facility, conciseness, and great extent of the method of computation, or algebraic part. It is for the sake of these advantages that so many symbols are employed in algebra. . . . . It [algebra] may have been employed to cover, under a complication of symbols, obstruse doctrines, that could not bear the light so well in a plain geometrical form; but, without doubt, obscurity may be avoided in this art as well as in geometry, by defining clearly the import and use of the symbols, and proceeding with care afterwards.

"Hence[2] the $\sqrt{-1}$ or the square-root of a negative, implies an imaginary quantity, and in resolution is a mark or character of the impossible cases of a problem; unless it is compensated by another imaginary symbol or supposition, when the whole expression may have real signification. Thus $1+\sqrt{-1}$, and $1-\sqrt{-1}$ taken separately are imaginary, but their sum is 2; as the conditions that separately

[1] Colin Maclaurin, *A Treatise of Fluxions.* In Two Books, Vol. II (Edinburgh, MDCCXLII), p. 575, 576.

[2] C. Maclaurin, *op. cit.*, Vol. II, p. 577, 578.

would render the solution of a problem impossible, in some cases destroy each others effect when conjoined. . . . . The theorems that are sometimes briefly discovered by the use of this symbol, may be demonstrated without it by the inverse operation, or some other way; and tho' such symbols are of some use in the computations in the method of fluxions, its evidence cannot be said to depend upon any arts of this kind."

718. *Charles Babbage* stresses the power of algebraic symbolism as follows:[1] "The quantity of meaning compressed into small space by algebraic signs, is another circumstance that facilitates the reasonings we are accustomed to carry on by their aid. The assumption of lines and figures to represent quantity and magnitude, was the method employed by the ancient geometers to present to the eye some picture by which the course of their reasonings might be traced: it was however necessary to fill up this outline by a tedious description, which in some instances even of no peculiar difficulty became nearly unintelligible, simply from its extreme length: the invention of algebra almost entirely removed this inconvenience. . . . . A still better illustration of this fact is noticed by Lagrange and Delambre, in their report to the French Institute on the translation of the works of Archimedes by M. Peyrard.[2] It occurs in the ninth proposition of the second book on the equilibrium of planes, on which they observe, 'La demonstration d'Archimede a trois énormes colonnes in-folio, et n'est rien moin que lumineuse.' Eutochius commence sa note 'en disant que le theorême est fort peu clair, et il promet de l'expliquer de son mieux. Il emploie quatre colonnes du même format et d'un caractère plus serré sans reussir d'avantage; au lieu que quatre lignes d'algebre suffisent a M. Peyrard pour mettre la verité du theorême dans le plus grand jour.' "

But Babbage also points out the danger of lack of uniformity in notations: "Time which has at length developed the various bearings of the differential calculus, has also accumulated a mass of materials of a very heterogeneous nature, comprehending fragments of unfinished theories, contrivances adapted to peculiar purposes, views perhaps sufficiently general, enveloped in notation sufficiently obscure, a multitude of methods leading to one result, and bounded by the same difficulties, and what is worse than all, a profusion of notations (when we regard the whole science) which threaten, if not duly

[1] C. Babbage, "On the Influence of Signs in Mathematical Reasoning," *Trans. Cambridge Philosophical Society*, Vol. II (1827), p. 330.

[2] *Ouvrages d'Archimede* (traduites par M. Peyrard), Tom. II, p. 415.

corrected, to multiply our difficulties instead of promoting our progress," p. 326.

719. *E. Mach*[1] discusses the process of clarification of the significance of symbols resulting from intellectual experimentation, and says: "Symbols which initially appear to have no meaning whatever, acquire gradually, after subjection to what might be called intellectual experimenting, a lucid and precise significance. Think only of the negative, fractional and variable exponents of algebra, or of the cases in which important and vital extensions of ideas have taken place which otherwise would have been totally lost or have made their appearance at a much later date. Think only of the so-called imaginary quantities with which mathematicians long operated, and from which they even obtained important results ere they were in a position to assign to them a perfectly determinate and withal visualizable meaning. . . . . Mathematicians worked many years with expressions like $\cos x+\sqrt{-1}\sin x$ and with exponentials having imaginary exponents before in the struggle for adapting concept and symbol to each other the idea that had been germinating for a century finally found expression [in 1797 in Wessel and] in 1806 in Argand, viz., that a relationship could be conceived between magnitude and *direction* by which $\sqrt{-1}$ was represented as a mean direction-proportional between $+1$ and -1."

720. *B. Branford* expresses himself on the "fluidity of mathematical symbols" as follows:[2] "Mathematical symbols are to be temporarily regarded as rigid and fixed in meaning, but in reality are continually changing and actually fluid. But this change is so infinitely gradual and so wholly subconscious in general that we are not sensibly inconvenienced in our operations with symbols by this paradoxical fact. Indeed, it is actually owing to this strange truth that progress in mathematical science is possible at all. An excellent instance is the gradual evolution of algebra from arithmetic—a clear hint this for teachers."

721. *A. N. Whitehead*[3] writes on the power of symbols: "By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race. Before the introduction of the

[1] E. Mach, *Space and Geometry* (trans. T. J. McCormack, 1906), p. 103, 104.

[2] Benchara Branford, *A Study of Mathematical Education* (Oxford, 1908), p. 370.

[3] A. N. Whitehead, *An Introduction to Mathematics* (New York and London, 1911), p. 59.

Arabic notation, multiplication was difficult, and the division even of integers called into play the highest mathematical faculties. Probably nothing in the modern world would have more astonished a Greek mathematician than to learn that, under the influence of compulsory education, the whole population of Western Europe, from the highest to the lowest, could perform the operation of division for the largest numbers. . . . . Mathematics is often considered a difficult and mysterious science, because of the numerous symbols which it employs. Of course, nothing is more incomprehensive than a symbolism which we do not understand. Also a symbolism, which we only partially understand and are unaccustomed to use, is difficult to follow. . . . . By the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain. It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them."

722. *H. F. Baker*[1] discusses the use of symbols in geometry: "Two kinds of symbols are employed in what follows: Symbols for geometrical elements, generally points, which we may call element-symbols; and symbols subject to laws of computation, which we may call algebraic symbols. . . . . The use of algebraic symbols for geometrical reasoning dates from very early times, ... and has prompted much in the development of Abstract Geometry. . . . . And, while it would be foolish not to employ the symbols for purposes of discovery, the view taken in the present volume is, that the object of a geometrical theory is to reach such a comprehensive scheme of conceptions, that every geometrical result may be obvious on geometrical grounds alone."

723. *H. Burkhardt*[2] utters his conviction that in geometry symbolic language has limitations: "An all-inclusive geometrical symbolism, such as Hamilton and Grassmann conceived of, is impossible."

724. *P. G. Tait*[3] emphasizes the need of self-restraint in the pro-

[1] H. F. Baker, *Principles of Geometry*, Vol. I (Cambridge, 1922), p. 70.

[2] H. Burkhardt in *Jahresbericht der Deutschen Math. Vereinigung*, Vol. V (1901, p. 52), p. xi.

[3] P. G. Tait, *An Elementary Treatise on Quaternions* (2d ed., enlarged; Cambridge. Preface dated 1873), p. xi.

posal of new notations: "Many abbreviations are possible, and sometimes very useful in private work; but, as a rule, they are unsuited for print. Every analyst, like every shorthand writer, has his own special contractions; but when he comes to publish his results, he ought invariably to put such devices aside. If all did not use a common mode of public expression, but each were to print as he is in the habit of writing for his own use, the confusion would be utterly intolerable."

O. S. Adams, in a review of W. D. Lambert's *Figure of the Earth*,[1] dwells upon the notational diversities in geodesy:[2] "A feature of the publication that is to be commended is the table of comparison of symbols used by various authors. One of the things that is always discouraging in reading scientific literature is the use of different notations for the same thing by different authors. Not the least of such offenders against clearness are the mathematicians both pure and applied. This table is therefore of great value to those who want to look into this question without too much preliminary study."

725. *British Committee*.[3]—The attention which should be given to typography is brought out by a British Committee as follows: "The process of 'composition' of ordinary matter consists in arranging types uniform in height and depth (or 'body' as it is termed) in simple straight lines. The complications peculiar to mathematical matter are mainly of two kinds.

"First, figures or letters, generally of a smaller size than those to which they are appended, have to be set as indices or suffixes; and consequently, except when the expressions are of such frequent occur-

[1] W. D. Lambert, *Effect of Variations in the Assumed Figure of the Earth on the Mapping of a Large Area*, Spec. Pub. No. 100 of the U.S. Coast and Geodetic Survey (Washington, 1924), p. 26, 35.

[2] O. S. Adams, *Science* (N.S.), Vol. LX (1924), p. 269.

[3] "Report of the Committee, consisting of W. Spottiswoode, F.R.S., Professor Stokes, F.R.S., Professor Cayley, F.R.S., Professor Clifford, F.R.S., and J. W. L. Glaisher, F.R.S., appointed to report on Mathematical Notation and Printing with the view of leading Mathematicians to prefer in optional cases such forms as are more easily put into type, and of promoting uniformity of notation," from the *British Association Report for 1875*, p. 337.

"Suggestions for notation and printing" were issued in 1915 by the Council of the London Mathematical Society (see *Mathematical Gazette*, Vol. VIII [London, 1917], p. 172, 220) and by G. Peano in an article, "L'esecuzione tipografica delle formule matematiche," in *Atti della r. accademia delle scienze di Torino*, Vol. LI (Torino, 1916), p. 279–86. See also Gino Loria, "I doveri dei simboli matematici," *Bollettino della Mathesis* (Pavia, 1916), p. 32–39.

rence as to make it worth while to have them cast upon type of the various bodies with which they are used, it becomes necessary to fit these smaller types in their proper positions by special methods. This process, which is called 'justification,' consists either in filling up the difference between the bodies of the larger and smaller types with suitable pieces of metal, if such exist, or in cutting away a portion of the larger, so as to admit the insertion of the smaller types.

"The second difficulty arises from the use of lines or 'rules' which occur between the numerator and denominator of fractions, and (in one mode of writing) over expressions contained under radical signs. In whatever part of a line such a rule is used, it is necessary to fill up, or compensate, the thickness of it throughout the entire line. When no letters or mathematical signs occur on a line with the rule the process is comparatively simple; but when, for example, a comma or sign of equality follows a fraction, or a $+$ or $-$ is prefixed to it, the middle of these types must be made to range with the rule itself, and the thickness of the rule must be divided, and half placed above and half below the type.

"The complications above described may arise in combination, or may be repeated more than once in a single expression; and in proportion as the pieces to be 'justified' become smaller and more numerous, so do the difficulties of the workman, the time occupied on the work, and the chances of subsequent dislocation of parts augment.

"The cost of 'composing' mathematical matter may in general be estimated at three times that of ordinary or plain matter. . . . .

"Your Committee are unwilling to close this Report without alluding to the advantages which may incidentally accrue to mathematical science by even a partial adoption of the modifications here suggested. Anything which tends towards uniformity in notation may be said to tend towards a common language in mathematics; and whatever contributes to cheapening the production of mathematical books must ultimately assist in disseminating a knowledge of the science of which they treat.

"MATHEMATICAL SIGNS NOT INVOLVING 'JUSTIFICATION'

$$\times \quad - \quad + \quad = \quad \surd \quad \pm \quad :: \quad \therefore \quad \because \quad : \quad \} \quad < \quad > \quad \div$$

$$\Big(\quad \Big[\quad \Big\} \quad \int \quad \sqrt{}$$

$$a,\ a_1,\ a^1,\ a^2,\ a_2\ \&\ ,\ a^{\frac{1}{2}},\ a_{\frac{1}{2}}$$

"EQUIVALENT FORMS

Involving Justification	Not Involving Justification
$\frac{x}{a}$	$x \div a$ or $x:a$
$\sqrt{x}$	$\sqrt{}x$ or $x^{\frac{1}{2}}$
$\sqrt[3]{x}$	$\sqrt[3]{}x$ or $x^{\frac{1}{3}}$
$\sqrt{x-y}$	$\sqrt{}(x-y)$ or $(x-y)^{\frac{1}{2}}$
$\sqrt{-1}$	i
$x.\overline{x+a}$	$x(x+a)$
$\epsilon^{\frac{n\pi x}{a}}$	exp $(n\pi x)(a)$
$\tan^{-1} x$	arc tan x
$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$	$x:y:z=l:m:n$"

726. *In the United States,* the editors of several mathematical journals,[1] in 1927, issued "Suggestions to Authors" for typewriting articles intended for print. The editors recommend the use of the solidus to avoid fractions in solid lines, the use of fractional exponents to avoid root signs; they give a list of unusual types available on the monotype machines which are used in printing the mathematical journals.

B. EMPIRICAL GENERALIZATIONS ON THE GROWTH OF MATHEMATICAL NOTATIONS

727. *Forms of symbols.*—A. *Primitive forms:* (1) Many symbols originated as abbreviations of words. Later, some of them (their ancestral connections being forgotten) assumed florescent forms and masqueraded as ideographic symbols. Examples: + (*et*), π (periphery of circle of unit diameter), $\int$ (*summa*), *d* (*differentia*), *i* (imaginary), log (logarithm), lim (limit), $\sim$ (similar), sin, cos, tan, sec. (2) Others are pictographic or picture symbols, such as $\triangle$ (triangle), $\bigcirc$ (circle), $\square$ (square), $\|$ (parallel), $\varnothing$ (parallelogram). (3) Others still are ideographic or arbitrary symbols: $\times$ (multiplication), $\div$ (division), ( ) aggregation, $\therefore$ (therefore), $\because$ (since), = (equality), and letters (not abbreviates) representing numbers or magnitudes.

[1] For instance, see *Bull. Amer. Math. Soc.*, Vol. XXXIII (May–June, 1927), the advertising page following p. 384.

B. *Incorporative forms,* representing combinations of two or more mathematical ideas and symbols, such as $\int_a^b$ in definite integrals, ${}_nC_r$ in the theory of combinations, $\lim\limits_{x \to a}$ in the theory of limits, $f(x)$ in the theory of functions.

C. The preference for symbols which avoid double or multiple lines of type $\left(\text{as illustrated by } \frac{a^n}{b^m}\right)$ is intensified by the decline of type-setting by hand and the increased use of machines.

Invention of symbols.—Whenever the source is known, it is found to have been individualistic—the conscious suggestion of one mind.

728. *Nature of symbols.*—(1) Some are merely shorthand signs which enable an otherwise long written statement to be compressed within a small space for convenient and rapid mental survey. (2) Others serve also in placing and keeping logical relationships before the mind. (3) Adaptability of symbols to changing viewpoint and varying needs constitutes superiority. The Leibnizian $\int$ has admitted of incorporative devices in integration, such as the Newtonian $\square$ or $\acute{x}$ could not admit so easily. The first derivative $\frac{dy}{dx}$ lends itself readily to change into differentials by elementary algebraic processes.

729. *Potency of symbols.*—Some symbols, like a^n, $\frac{d^ny}{dx^n}$, $\sqrt{n}$, $\log n$, that were used originally for only positive integral values of n stimulated intellectual experimentation when n is fractional, negative, or complex, which led to vital extensions of ideas.

730. *Selection and spread of symbols.*—(1) While the origin of symbols has been individualistic, their adoption has been usually by nonconferring groups of mathematicians. As in other inventions, so here, many symbols are made, but few are chosen. The list of discarded mathematical signs would fill a volume. (2) In no branch of mathematics is the system of symbols now used identical with what any one inventor ever designed them to be; present notations are a mosaic of individual signs of rejected systems. Modern elementary algebra contains symbols from over a dozen different inventors. Excepting Leibniz and Euler, no mathematician has invented more than two ideographs which are universally adopted in modern mathematics. (3) Often the choice of a particular symbol was due to a special configuration of circumstances (large group of pupils friendships, popularity of a certain book, translation of a text) other than

those of intrinsic merit of the symbol. Thus the inferior : : was adopted in proportion after = had been introduced as the sign of equality. At times there existed other extraneous influences. Perhaps commercial relations with Hebrews and Syrians led the Greeks to abandon the Herodianic numerals and to adopt an inferior alphabetic numeral system. Friction between British and continental mathematicians prevented England, to her loss, from adopting in the eighteenth century the Leibnizian calculus notations, in which the great analytical researches of Swiss and French mathematicians were being recorded. (4) Mathematical symbols cross linguistic borders less readily than do mathematical ideas. This observation applies to abbreviates and ideographs rather than to pictographs. The Arabs adopted no symbols used by the Babylonians, Greeks, or Hindus (except probably the Hindu numerals), nevertheless the Arabs acquired the Babylonian sexagesimals, Hindu trigonometry, part of Hindu algebra, Diophantine algebra, and the Greek system of geometric axioms and proofs. The Arabic algebraic symbolism of the fifteenth century antedated the German *coss*, yet acquired no foothold in Christian Europe. In the sixteenth century the Italian solutions of cubic and quartic equation were eagerly studied in Germany, but the Italian notation was ignored by the Germans. Sometimes even national boundaries between linguistically homogeneous peoples yield diversity of symbols. The notation 2.5 means $2\frac{5}{10}$ in the United States and 2 times 5 in Great Britain; the notation $2\cdot 5$ means 2 times 5 in the United States and $2\frac{5}{10}$ in Great Britain. The symbols + and − which sprang up in Germany in the fifteenth century were extremely convenient, yet they were not introduced into France, Italy, England, and Spain until after the middle of the sixteenth century. These symbols first reached France by the republication in Paris, in 1551, of an algebra written in Latin by the German author, Scheubel, and first published in Germany. These symbols first appear in Italy in 1608 in a book written by a German (Clavius) who had taken up his residence in Rome. They first appear in Spain in a book written in Spain in 1552 by a German immigrant. In some cases the symbols were actually carried across national borders by individual travelers. The book trade did not seem to stimulate the prompt spread of symbols. The continent of America offers the earliest fully developed number system (the Maya system), constructed on the principle of local value and a symbol for zero, but the knowledge of this did not spread at the time. Mathematical symbols proposed in America such as $\Bumpeq$ for equivalent or $\doteq$ for approach to a limit, or Gibbs's symbols in vector analysis, or

C. S. Peirce's signs in symbolic logic, were not adopted in Europe. In fact, no American mathematical symbol, except perhaps the dollar mark ($), has found acceptance in Europe. And yet the ideas set forth in Gibbs's vector algebra and in C. S. Peirce's symbolic logic admittedly impressed themselves upon European thought. (5) The rejection of old symbols and the adoption of new ones are greatly retarded by acquired habits. The symbol $\div$ for subtraction has persisted for four hundred years among small groups of Teutonic writers even though less desirable than the usual symbol $-$. (6) Mathematicians have uniformly resisted the general adoption of very lavish and profuse symbolism for any branch of mathematics. The eruption of symbols due to Oughtred, Hérigone, Rahn, and Hindenburg has been followed by the adoption of only certain few of them.

731. *State of flux.*—The maxim of Heraclitus that all things are in a state of flux applies to mathematical symbolism. During the last four centuries symbols of aggregation have undergone several changes. The : : in proportion is now passing into disuse; the arrow as a symbol for approaching a limit is meeting with enthusiastic adoption everywhere.

732. *Defects in symbolism.*—(1) Sometimes the haphazard selection by non-conferring groups of workers has led to two or more things being represented by the same symbol. One cannot interpret dx without first knowing whether it occurs in algebra or in the calculus. The sign $\sim$ may mean "difference" or "similar to" or "equivalent to" or "approximately equal to" or, in the case of $a_v \sim b_v$, that the ratio of a_v and b_v approaches a constant as $v \to \infty$. (2) Often two or more symbols are used for the same thing, causing waste of intellectual effort. Algebra has a calculus of radicals and a calculus of fractional exponents, both covering the same field. Millions of children have been forced to master both systems. Here Descartes and Newton missed a splendid opportunity to prevent useless duplication. However, there are fields where a certain amount of duplication is convenient and not burdensome; for instance, in forms of derivatives, $\frac{dy}{dx}$, $f'(x)$, also $\dot{x}$ for $\frac{dx}{dt}$.

733. *Individualism a failure.*—As clear as daylight is the teaching of history that mathematicians are still in the shadow of the Tower of Babel and that individual attempts toward the prompt attainment of uniformity of notations have been failures. Mathematicians have not been profiting by the teachings of history. They have failed to adopt

the alternative procedure. Mathematical symbols are for the use of the mathematical community, and should therefore be adopted by that community. The success of a democracy calls for mass action. As long as it is not physically possible for all to gather in close conference, the political procedure of acting through accredited representatives becomes necessary. Mathematicians as a group are far less efficient than are the citizens of our leading republics as groups. The latter have learned to act through representatives; they have learned to subordinate their individual preferences and to submit themselves to the laws passed for the greatest good of all. The mathematicians have not yet reached that stage of co-operative endeavor in the solution of the all-important question of mathematical symbolism. There is need of cultivation of the spirit of organization and co-operation.

C. CO-OPERATION IN SOME OTHER FIELDS OF SCIENTIFIC ENDEAVOR

734. *Electric units.*—How does it happen that in electricity and magnetism there is perfect agreement in the matter of units of measurement in scientific books and in commercial affairs throughout the world? How was this uniformity brought about? The answer is that the scientists in this field of investigation held international meetings, that representatives of the various countries in serious conference were able to reach agreements which the separate countries afterward confirmed by legal enactments. At the International Congress of Electricians, held in Paris in 1881, the centimeter-gram-second system was adopted and the foundation was laid for the adoption also of commercial units. A uniform system was reached only by international conferences, and by the furtherance of the spirit of compromise. The millimeter-second-milligram system of Wilhelm Weber was dropped; the British Association unit of resistance was dropped. In their stead appeared slightly modified units.

It must be admitted that in the selection of electrical and magnetic units forces were at work which are not present to the same degree in attempts to unify mathematical notations. In the case of electromagnetism there existed financial considerations which made unification imperative. International commerce in electrical machinery and electric instruments rendered a common system of units a necessity. Without international means of measurement, international exports and imports would be seriously embarrassed.

735. *Star Chart and Catalogue.*—That international co-operation on the part of scientific men is possible is shown again by the gigantic

enterprise to prepare an *International Star Chart and Catalogue,* which was initiated by the International Astrographic Congress that met at Paris in 1887. About eighteen observatories in different parts of the world entered upon this colossal task. In this undertaking the financial aspect did not play the intense rôle that it did in securing uniformity of electrical units. It constitutes an example of the possibility of international co-operation in scientific work even when material considerations are not involved.

It is true, however, that in this astronomical program the individual worker encountered fewer restrictions than would a mathematician who was expected in printed articles to follow the recommendations of an international committee in matters of mathematical notation.

D. GROUP ACTION ATTEMPTED IN MATHEMATICS

736. It is only in recent years that attempts have been made to introduce uniform notations through conferences of international representatives and recommendations made by them. The success of such endeavor involves two fundamental steps. In the first place, the international representatives must reach an agreement; in the second place, individual writers in mathematics must be willing to adopt the recommendations made by such conferences. Thus far no attempt at improvement of notations can be said to have weathered both of these ordeals.

737. *Vector analysis.*—In 1889 a small international organization was formed, called the "International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics." Sir Robert S. Ball was president and A. Macfarlane secretary. At the international congress of mathematicians held at Rome in 1908 a committee was appointed on the unification of vectorial notations, but at the Congress held in Cambridge in 1912, no definite conclusion had been reached (§ 509). Considerable discussion was carried on in various journals, some new symbols were proposed, but the final and critical stage when a small representative group of international representatives gather around a table and make serious attempts to secure an agreement was never reached. The mathematical congresses held at Strasbourg in 1920 and at Toronto in 1924 did not take up the study of vectorial notations. The Great War seriously interrupted all plans for co-operation in which all the great nations could participate.

738. *Potential and elasticity.*—Another failure, due chiefly to the break of friendly international relations during the Great War,

marks the work of an International Committee of sixty members, formed in 1913, on the unification of notations and terminology in the theories of potential and elasticity. Arthur Korn, of Charlottenburg, was corresponding secretary. The Committee had laid out a regular program extending over about eight years. Mathematicians, physicists, and astronomers of the world were asked to co-operate and, in the first place, to assist the Committee by answering the question, "What are the notions and notations in respect to which it is desirable to establish uniformity?" In course of the year 1914 a second circular was to be issued for suggestions as to methods by which the desired unification may be brought about. It was the original intention to send out a third circular in 1916, setting forth the points of dispute, and to arrange a discussion thereon at the International Congress of Mathematicians that was to be held in 1916. A fourth circular, to be issued in 1917, should contain a report of the discussion and provide an opportunity for those who should not have been present at the Congress to express their views in writing. All the proposals and contributions to the discussion were to be sifted and arranged, and a fifth circular was to be sent out in 1919 containing a statement of the points with regard to which agreement had been reached and to take a vote on those in dispute. The final vote was to be taken at the International Congress of Mathematicians in 1920. The year following, a sixth circular was to announce the result of the vote. This extensive scheme was nipped in the bud by the war. The present writer has seen only the first circular issued by the Committee.[1] As no truly international congress of mathematicians has been held since 1912, the proposed work of unification has not yet been resumed. The old organization no longer exists. When the work is started again, it must be *de novo*.

739. *Actuarial science.*—A third movement, which was not interrupted by war, met with greater success. We refer to the field of actuarial science. In this line the British Actuarial Society had been active for many years, devising a full system of notation, so clear and efficient as to meet with quite general approval. With some slight modifications, the English proposals were adopted by the International Congress of Actuaries in 1895 and 1898. As a result of this action, many of the symbols in a very long list have been widely adopted, but as far as we can learn, they are not used to the exclusion of all other symbols. This movement marks a partial success. Per-

[1] For an account of the movement see *Isis*, Vol. I (1913), p. 491; Vol. II (1914), p. 184.

haps some improvement in the mode of procedure might have led to a more universal acceptance of the symbols agreed upon by the international experts. Perhaps a mistake was made in endeavoring to reach agreement on too many symbols at one time. The general adoption of a small number of symbols concerning which there was the greatest unanimity, rather than a list extending over six solid pages of printed matter, might have been better.[1]

Of the three attempts at unification through international action which we have cited, the first two failed, because the first essential step was not taken; the international representatives did not reach agreement. In the third experiment agreement was reached, but, as far as can be determined, more recent individual workers of different countries have not felt themselves compelled to adopt the symbols agreed upon. In fact, the diversity of plans of insurance which have been adopted in different countries makes it often difficult for some actuaries on the European Continent to use the notation which had its origin in England.

E. AGREEMENTS TO BE REACHED BY INTERNATIONAL COMMITTEES THE ONLY HOPE FOR UNIFORMITY IN NOTATIONS

740. Uniformity of mathematical notations has been a dream of many mathematicians—hitherto an iridescent dream. That an Italian scientist might open an English book on elasticity and find all formulae expressed in symbols familiar to him, that a Russian actuary might recognize in an English text signs known to him through the study of other works, that a German physicist might open an American book on vector analysis and be spared the necessity of mastering a new language, that a Spanish specialist might relish an English authority on symbolic logic without experiencing the need of preliminary memorizing a new sign vocabulary, that an American traveling in Asiatic Turkey might be able to decipher, without the aid of an interpreter, a bill made out in the numerals current in that country, is indeed a consummation devoutly to be wished.

Is the attainment of such a goal a reasonable hope, or is it a utopian idea over which no mathematician should lose precious time? In dealing with perplexing problems relating to human affairs we are prone to look back into history, to ascertain if possible what

[1] See list of symbols in the *Transactions of the Second International Actuarial Congress*, held at London in 1898 (London, 1899), p. 618–24.

light the past may shed upon the future and what roads hitherto have been found impassable. It is by examining the past with an eye constantly on the future that the historical student hopes to make his real contribution to the progress of intelligence.

741. The admonition of history is clearly that the chance, haphazard procedure of the past will not lead to uniformity. The history of mathematical symbolism is characterized by a certain painting representing the landing of Columbus; the artist painted three flags tossed by the breeze—one east, one west, and one south—indicating a very variable condition of the wind on that memorable day. Mathematical sign language of the present time is the result of many countercurrents. One might think that a perfect agreement upon a common device for marking decimal fractions could have been reached in the course of the centuries.

No doubt many readers will be astonished to hear that in recent books, printed in different countries, one finds as many as nine different notations for decimal fractions (§ 286). O Goddess of Chaos, thou art trespassing upon one of the noblest of the sciences! The force of habit conspires to the perpetuity of obsolete symbols. In Descartes' *Geometry* of 1637 appeared the modern exponential notation for positive integral powers. Its ideal simplicity marked it as a radical advance over the older fifteenth- and sixteenth-century symbolisms. Nevertheless, antiquated notations maintained their place in many books for the following fifty or seventy-five years. Other cases similar to this could be cited. In such redundancy and obsoleteness one sees the hand of the dead past gripping the present and guiding the future.

742. Students of the history of algebra know what a struggle it has been to secure even approximate uniformity of notation in this science, and the struggle is not yet ended. The description of all the symbols which have been suggested in print for the designation of the powers of unknown quantities and also of known quantities is a huge task. Rival notations for radical expressions unnecessarily complicate the study of algebra in our high schools today. And yet ordinary algebra constitutes the gateway to elementary mathematics. Rival notations embarrass the beginner at the very entrance into the field of this science.

This confusion is not due to the absence of individual efforts to introduce order. Many an enthusiast has proposed a system of notation for some particular branch of mathematics, with the hope, perhaps, that contemporary and succeeding workers in the same field

would hasten to adopt his symbols and hold the originator in grateful remembrance. Oughtred in the seventeenth century used over one hundred and fifty signs for elementary mathematics—algebra, geometry, and trigonometry. Many of them were of his own design. But at the present time his sign of multiplication is the only one of his creation still widely used; his four dots for proportion are redundant and obsolescent; his sign for arithmetical difference ($\backsim$) has been employed for wholly different purposes. In France, Hérigone's writings of the seventeenth century contained a violent eruption of symbols, now known only to the antiquarian. The Hindenburg combinatorial school at the close of the eighteenth century in Germany introduced complicated symbolic devices which long since passed into innocuous desuetude. The individual designer of mathematical symbols, looking forward to enduring fame, is doomed to disappointment. The direction of movement in mathematical notation, when left to chance, cannot be predicted. When we look into the history of symbolism, we find that real merit has not always constituted the determining factor. Frequently wholly different circumstances have dominated the situation. The sign of division ($\div$) now used in England and America first appears in a Swiss algebra of 1659 (§ 194), written in the German language, which enjoyed no popularity among the Germans. But an English translation of that book attracted wide attention in England and there led to the general adoption of that sign for division. The outcome of these chance events was that Great Britain and America have at the present time a sign of division used nowhere else; the Leibnizian colon employed for division in Continental Europe is not used for that purpose in the English-speaking countries. Another example of the popularity of a book favoring the adoption of a certain symbol is Byerly's *Calculus*, which secured the adoption in America of J. E. Oliver's symbol ($\doteq$) for approach to a limit. Oughtred's pupils (John Wallis, Christopher Wren, Seth Ward, and others) helped the spread in England of Oughtred's notation for proportion which attained popularity also in Continental Europe. The Dutch admirers of Descartes adopted his sign of equality which became so popular in Continental Europe in the latter part of the seventeenth century as almost to force out of use the Recordian sign of equality now generally adopted. The final victory of the Recordian sign over the Cartesian rival appears to be due to the fact that Newton and Leibniz, who were the two brightest stars in the mathematical firmament at the beginning of the eighteenth century, both used in print the Recordian equality. Thus history

points to the generalization that individual effort has not led to uniformity in mathematical language. In fact, the signs of elementary algebra constitute a mosaic composed of symbols taken from the notations of more than a dozen mathematicians.

743. In the various fields of mathematics there has been, until very recently, no method of selection, save through mere chance. Consequently, some symbols have several significations; also, the same idea is often designated by many different signs. There is a schism between the situation as it exists and what all workers would like it to be. Specialists in advanced fields of mathematics complain that the researches of others are difficult to master because of difference in sign language. Many times have we called attention to passages in the writings of specialists in modern mathematics, deploring the present status. Usually the transition from one notation to another is not easy. In the eighteenth century mathematicians framed and tried to remember a simple rule for changing from the Leibnizian dx to the Newtonian $\dot{x}$. But the rule was inaccurate, and endless confusion arose between differentials and derivatives.

744. One might think that such a simple matter as the marking of the hyperbolic functions would not give rise to a diversity of notation, but since the time of Vincenzo Riccati there have been not less than nine distinct notations (§§ 529, 530, 536). No better is our record for the inverse trigonometric functions. For these there were at least ten varieties of symbols (§§ 532–36). Two varieties, namely, John Herschel's sign looking like exponent minus one, as in $\sin^{-1} x$, and the "arc sin," "arc cos," etc., of the Continent, still await the decision as to which shall be universally adopted.

745. The second half of the eighteenth century saw much experimentation in notations of the calculus, on the European Continent. There was a great scramble for the use of the letters D, d, and δ. The words "difference," "differential," "derivative," and "derivation" all began with the same letter. A whole century passed before any general agreement was reached. Particular difficulty in reaching uniformity prevailed in the symbolism for partial derivatives and partial differentials. A survey has revealed thirty-five different varieties of notation for partial derivatives (§§ 593–619). For about one hundred and fifty years experimentation was carried on before some semblance of uniformity was attained. Such uniformity could have been reached in fifteen years, instead of one hundred and fifty, under an efficient organization of mathematicians.

746. In this book we have advanced many other instances of

"muddling along" through decades, without endeavor on the part of mathematicians to get together and agree upon a common sign language. Cases are found in descriptive geometry, modern geometry of the triangle and circle, elliptic functions, theta functions, and many other fields. An investigator wishing to familiarize himself with the researches of other workers in his own field must be able not only to read several modern languages, English, French, Italian, and German, but he must master also the various notations employed by different authors. The task is unnecessarily strenuous. A French cynic once said: "Language is for the suppression of thought." Certainly the diversity of language tends to the isolation of thought. No doubt many mathematicians of the present time are wondering what will be the fate of the famous symbolisms of Peano and of Whitehead and Russell. In the light of past events we predict that these systems of symbols will disintegrate as did the individual systems of the seventeenth and eighteenth centuries.

747. It cannot be denied that in the desire to attain absolute logical accuracy in complicated and abstract situations arising in the study of the foundations of mathematics an elaborate system of ideographs such as are used by Peano and especially by Whitehead and Russell has been found serviceable (§§ 688–92, 695). But these symbols are intended by their inventors to be used in the expression of all mathematics in purely ideographic form.[1] Thus far, such a system designed to take the place of ordinary language almost entirely has not been found acceptable to workers in the general field of mathematics.[2] All teaching of history discourages attempts to force a scheme of this sort upon our science. More promising for mathematicians in general is some plan like that of E. H. Moore. Though fully cognizant of the need of new symbols for our growing science, he uses only a restricted amount of symbolism (§ 694) and aims at simplicity and flexibility of the signs used. He aims at a system of logical signs that would be acceptable to working mathematicians. New symbols should be ushered in only when the need for them is unquestioned. Moore's program for a mathematical sign

[1] G. Peano, *Formulaire de mathématiques*, Tom. I (Turin, 1895), Introduction, p. 52; A. N. Whitehead and B. Russell, *Principia mathematica*, Vol. I (Cambridge, 1910), Preface; also Vol. III (1913), Preface.

[2] On this question see also A. Voss, *Ueber das Wesen der Mathematik* (Leipzig und Berlin, 2. Aufl., 1913), p. 28; A. Padoa, *La logique déductive* (Paris, 1912); H. F. Baker, *Principles of Geometry*, Vol. I (Cambridge, 1922), p. 62, 137, and Vol. II, p. 4, 16, 153.

language is worthy of serious study. He believes in international co-operation and agreement, in matters of notation.

748. In the light of the teaching of history it is clear that new forces must be brought into action in order to safeguard the future against the play of blind chance. The drift and muddle of the past is intolerable. We believe that this new agency will be organization and co-operation. To be sure, the experience of the past in this direction is not altogether reassuring. We have seen that the movement toward uniformity of notation in vector analysis fell through. There was a deep appreciation, on the part of the individual worker, of the transcendent superiority of his own symbols over those of his rivals. However, this intense individualism was not the sole cause of failure. Eventually, an appreciation of the ludicrous might have saved the situation, had it not been for the disorganization, even among scientific men, resulting from the Great War. Past failures have not deterred our National Committee on Mathematical Requirements from recommending that their report on symbols and terms in elementary mathematics be considered by the International Congress of Mathematicians. The preparation of standardized symbols and abbreviations in engineering is now entered upon, under the sponsorship of the American Association for the Advancement of Science and other societies.

749. The adoption at a particular congress of a complete symbolism intended to answer all the present needs of actuarial science finds its counterpart in the procedure adopted in world-movements along somewhat different lines. In 1879 a German philologian invented a universal language, called "volapuk." Eight years later a Russian invented "esperanto," containing 2,642 root-words. Since then about eight modified systems of universal language have been set up. It will be recognized at once that securing the adoption of a universal spoken and written language by all intellectual workers is an infinitely greater task than is that of a universal sign symbolism for a very limited group of men called mathematicians. On the other hand, the task confronting the mathematician is more difficult than was that of the electrician; the mathematician cannot depend upon immense commercial enterprises involving large capital to exert a compelling influence such as brought about the creation and adoption of a world-system of electric and magnetic units. Even the meteorologist has the advantage of the mathematician in securing international signs; for symbols describing the weather assist in the quick spreading of predictions which may safeguard shipping, crops, and cattle.

The movement for a general world-language is laboring under two *impedimenta* which the mathematicians do well to avoid. In the first place, there has been no concerted, united action among the advocates of a world-language. Each works alone and advances a system which in his judgment transcends all others. He has been slow to learn the truth taught to the savage by his totem, as related by Kipling, that

"There are nine and sixty ways
Of constructing tribal lays
And every single one of them is right."

Unhappily each promoter of a world-language has been opposing all other promoters. The second mistake is that each system proposed represents a finished product, a full-armed Minerva who sprung from the head of some Jupiter in the realm of philology.

In the endeavor to secure the universal adoption of mathematical symbolism international co-operation is a *sine qua non.* Agreements by representatives must take the place of individual autocracy. Perhaps scientists are in a better position to cultivate internationalism than other groups.

The second warning alluded to above is that no attempt should be made to set up at any one time a full system of notation for any one department of higher mathematics. This warning has remained unheeded by some contemporary mathematicians. In a growing field of mathematics the folly of such endeavor seems evident. Who is so far sighted as to be able to foresee the needs of a developing science? Leibniz supplied symbols for differentiation and integration. At a later period in the development of the calculus the need of a symbolism for partial differentiation arose. Later still a notation for marking the upper and lower limits in definite integrals became desirable, also a mode of designating the passage of a variable to its limit. In each instance numerous competing notations arose and chaos reigned for a quarter- or a half-century. There existed no international group of representatives, no world-court, to select the best symbols or perhaps to suggest improvements of its own. The experience of history suggests that at any one congress of representative men only such fundamental symbols should be adopted as seem imperative for rapid progress, while the adoption of symbols concerning which there exists doubt should be discussed again at future congresses. Another advantage of this procedure is that it takes cognizance of a disinclination of mathematicians as a class to master the meaning of a symbol

and use it, unless its introduction has become a practical necessity. Mathematicians as a class have been accused of being lethargic, easily pledged to routine, suspicious of innovation.

Nevertheless, mathematics has been the means of great scientific achievement in the study of the physical universe. Mathematical symbols with all their imperfections have served as pathfinders of the intellect. But the problems still awaiting solution are doubtless gigantic when compared with those already solved. In studying the microcosm of the atom and the macrocosm of the Milky Way, only mathematics at its best can be expected to overcome the obstacles impeding rapid progress. The dulling effect of heterogeneous symbolisms should be avoided. Thus only can mathematics become the Goliath sword enabling each trained scientist to say: "There is none like that; give it me."

750. The statement of the Spottiswoode Committee bears repeating: "Anything which tends toward uniformity in notation may be said to tend towards a common language in mathematics, and .... must ultimately assist in disseminating a knowledge of the science of which they treat." These representative men saw full well that God does not absolve the mathematician from the need of the most economic application of his energies and from indifference to the well-tried wisdom of the ages.

Our mathematical sign language is still heterogeneous and sometimes contradictory. And yet, whatever it lacks appears to be supplied by the spirit of the mathematician. The defect of his language has been compensated by the keenness of his insight and the sublimity of his devotion. It is hardly worth while to indulge in speculation as to how much more might have been achieved with greater symbolic uniformity. With full knowledge of the past it is more to the point to contemplate the increasingly brilliant progress that may become possible when mathematicians readdress themselves to the task of breaking the infatuation of extreme individualism on a matter intrinsically communistic, when mathematicians learn to organize, to appoint strong and representative international committees whose duty it shall be to pass on the general adoption of new symbols and the rejection of outgrown symbols, when in their publications mathematicians, by a gentlemen's agreement, shall abide by the decisions of such committees.

ALPHABETICAL INDEX

(Numbers refer to paragraphs)

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